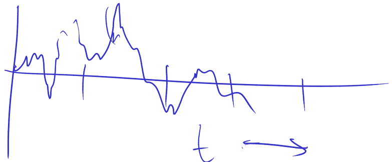
2

Last time : $W_t = \lim_{N \to \infty} W^N_{\lfloor Nt \rfloor}$ $\leftarrow$ Brownian motion.

$EX_k = 0$, $EX_k^2 = 1$, $X_k \to$ iid. $= \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{1}^{\lfloor Nt \rfloor} X_k$.



$t \in \mathbb{R}$.

Definition 8.18. We say a random variable Y is $\mathcal{F}_t$ measurable if $Y = \lim_{n\to\infty} f_n(W_{t_1}, \dots, W_{t_n})$ where $t_i \leqslant t$ for all i.

Definition 8.19. If $Y = \underbrace{f(W_{t_1}, \dots, W_{t_n})}$ for some function f and $0 \leqslant t_1 \cdots < t_n$, define $\boxed{\boldsymbol{E}_t Y} = \lim_{N\to\infty} \tilde{\boldsymbol{E}}_{\lfloor Nt \rfloor} f(W^N_{\lfloor Nt_1 \rfloor}, \dots, W^N_{\lfloor Nt_n \rfloor})$

Remark 8.20. $\boldsymbol{E}_t f(W_T) = u(t, W_t)$, where u is the function in Lemma 8.16.

Remark 8.21. The operator $\boldsymbol{E}_t$ satisfies the same properties as $\tilde{\boldsymbol{E}}_n$ (e.g. $\boldsymbol{E}_t(XY) = X\boldsymbol{E}_t Y$ if X is $\mathcal{F}_t$ measurable, independence lemma, etc.) These will be developed systematically in continuous time finance.

W^N, S_n, X_n etc

(int time)

$\lfloor Nt \rfloor$

Proposition 8.22. *W is a martingale.*

$E_t Y$ = conditional exp of Y given $\mathcal{F}_t$

$= E(Y \mid \mathcal{F}_t)$.

W

$(t \in \mathbb{R})$

E_t has the same properties as $\tilde{E}_n$ in the disc case

① X is $\mathcal{F}_t$ meas $\Rightarrow$ $E_t f(X) = f(X)$

② $\underline{X}$ is $\mathcal{F}_t$ meas & $\underline{Y \text{ is any thing}}$ $\Rightarrow$ $E_t(XY) = \underline{X E_t Y}$

③ X ind of $\mathcal{F}_t$ $\Rightarrow$ $E_t X = EX$

Prop: W is a Mg

Q: What does it mean for a cts time process to be a mg

Disc time def: M is a (disc time) mg if $\tilde{E}_n M_{n+1} = M_n$ $\forall n$.

$\to$ Cts time guess ①: $E_t(M_{t+1}) = M_t$ $\forall t$ (② Not good enough).

Def: We say M is a mg (cts time) if M is adapted and $\underline{E_s M_t = M_s}$ $\forall\, t \geq s$.

(Note: disc time: by tower prop & ind, $E_{n} M_{n+1} = M_n$ $\forall\, n$

$\iff E_m \underline{M_n} = \underline{M_m}$ $\forall\, n \geq m$)

<u>Prop</u>: W is a mg.

Pf: WTS $E_s W_t = W_s$ $\forall\, t \geq s$.

Note: $E_s W_t = E_s(W_t - W_s + W_s)$

$\boxed{\text{Trick \# 1 for B.M}}$

$= E_s(W_t - W_s) + E_s W_s$

$= E(W_t - W_s) + W_s$ ($\because W_t - W_s$ is ind of $\mathcal{F}_s$ & $W_t - W_s \sim N(0, t-s)$)

$= 0 + W_s = W_s$ ($\because W_s$ is $\mathcal{F}_s$ meas)

QED.

C^N

(int time).

$t \mapsto \lfloor Nt \rfloor$

8.4. Convergence of the Binomial Model.

(1) Let $r_N > -1$, and consider a bank that pays you interest r_N every $1/N$ time units.
(2) *Question:* Can we choose r_N so that this converges as $N \to \infty$.
(3) Let $C_0^N = 1$, $C_{n+1}^N = (1 + r_N)C_n^N$ and $C_t = \lim\limits_{N\to\infty} C^N_{\lfloor Nt \rfloor}$.

Proposition 8.23. *If* $r \in \mathbb{R}$, $r_N = r/N$, *then* $C_t = e^{rt}$.

Remark 8.24. Note $\partial_t C_t = rC_t$. The quantity r is known as the *continuously compounded interest rate.* $t \in \mathbb{R}$

Remark 8.25. If the interest rate is a constant r, then the discount factor is simply $\boxed{D_t = 1/C_t = e^{-rt}}$.

Pf: $r_N = \dfrac{r}{N}$ $\quad r \in \mathbb{R}$.

$$C^N_n = (1 + r_N)^n = \left(1 + \frac{r}{N}\right)^n.$$

$$\Rightarrow C^N_{\lfloor Nt \rfloor} = \left(1 + \frac{r}{N}\right)^{\lfloor Nt \rfloor} = \left[\underbrace{\left(1 + \frac{r}{N}\right)^N}\right]^{\frac{\lfloor Nt \rfloor}{N}}$$

Note $\lim\limits_{N \to \infty} \left(1 + \frac{r}{N}\right)^N = e^r$ & $\lim\limits_{N\to\infty} \dfrac{\lfloor Nt \rfloor}{N} = t$

$$\Rightarrow \lim_{N\to\infty} C^{N}_{\lfloor Nt \rfloor} = e^{rt}$$

ÖED

(1) Now consider the N period Binomial model, with parameters $0 < d_N < 1 + r_N < u_N$, with stock price denoted by S_n^N.
(2) Each time step for S^N denotes $1/N$ time units in real time. Can we chose u_N, d_N, r_N such that $S_t = \lim_{N\to\infty} S^N_{\lfloor Nt \rfloor}$ exists?
(3) Choose $r_N = r/N$, where $r \in \mathbb{R}$ is the continuously compounded interest rate.

doesn't dep on N.

Theorem 8.26. *Let $u, d > 0$ and choose*

$$u_N = 1 + \frac{r}{N} + \frac{u}{\sqrt{N}}, \qquad d_N = 1 + \frac{r}{N} - \frac{d}{\sqrt{N}}, \qquad \tilde{p} = \frac{d}{u+d}, \qquad \tilde{q} = \frac{u}{u+d}, \qquad \boxed{\sigma^2 = \tilde{p}u^2 + \tilde{q}d^2}\,.$$

Under the risk neutral measure, the processes $S^N_{\lfloor Nt \rfloor}$ converge weakly to $S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}$, where W is a Brownian motion. That is, for any bounded continuous function f,

GBM

$$\lim_{N\to\infty} \tilde{\boldsymbol{E}} f(S^N_{\lfloor Nt \rfloor}) = \tilde{\boldsymbol{E}} f(S_t) = \tilde{\boldsymbol{E}} f\Big(S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)\Big)$$

Remark 8.27. S_t above is called a *Geometric Brownian motion* with mean return rate r, and volatility σ.

Remark 8.28. The fact that we took the limit under the risk neutral measure is the reason the mean return rate r is the same as the interest rate r.

Remark 8.29. In this continuous time market you have the asset (whose price is denoted by S_t), and a bank with *continuously compounded* interest rate r (i.e. discount factor is $D_t = e^{-rt}$). You can trade *continuously* in time, and we are neglecting any transaction costs.

R.N. Prob. $\quad \tilde{p}_N = \dfrac{1+r_N - d_N}{u_N - d_N} = \dfrac{1+\frac{r}{N} - \left(1+\frac{r}{N} - \frac{d}{\sqrt{N}}\right)}{\left(1+\frac{r}{N}+\frac{u}{\sqrt{N}}\right) - \left(1+\frac{r}{N} - \frac{d}{\sqrt{N}}\right)}$

$$= \frac{d/\cancel{\sqrt{N}}}{(u+d)/\cancel{\sqrt{N}}} = \frac{d}{u+d}.$$

Theorem 8.30. *Consider a security that pays $f(S_T)$ at maturity time T. The arbitrage free price of this security at time t is given by*

$$\boxed{V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t\Big(D_T f(S_T)\Big) = \tilde{\boldsymbol{E}}_t\big(e^{-r(T-t)} f(S_T)\big)}$$

Proof. For the Binomial model we already know $V_n^N = \frac{1}{D_n^N}\tilde{\boldsymbol{E}}_n D^N_{\lfloor NT\rfloor} f(S^N_{\lfloor NT\rfloor})$. Set $n = \lfloor Nt \rfloor$ and send $N \to \infty$. □