HW 13 Q 3:

$$(X, Y) \sim N(\mu, \Sigma)$$

Q: $\mathrm{cor}(X, Y) = 0$? $\Longrightarrow$ X & Y are ind?

Q0: What does it mean for 2 d's RV's to be ind

X, Y discrete: $P(X = x_i, Y = y_i) = P(X = x_i) P(Y = y_i) \quad \forall \; x_i \in \mathrm{Range}(X), \; y_i \in \mathrm{Range}(Y)$

$\Longleftrightarrow X, Y$ ind $\quad (X, Y$ discrete$)$

X, Y cts $\longrightarrow$ $P(X = x) = 0 \quad \forall x \in \mathbb{R}$

$P(Y = y) = 0 \quad \forall y \in \mathbb{R}$

& $P(X = x, Y = y) = 0 \quad \forall x, y \in \mathbb{R}$

Q: $\forall A, B \subseteq \mathbb{R}$ (intervals) $P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$

this def of ind works for both disc & cts RV's.

Quick check X, Y are disc:

$$P(X \in A, Y \in B) = \sum_{x_i \in \text{Range}(X) \cap A} \sum_{y_j \in \text{Range}(Y) \cap B} P(X = x_i, Y = y_j)$$

$$\overset{\text{ind}\,\perp}{=} \underbrace{\left(\sum_{x_i \in \text{Range}(X)\cap A} P(X=x_i)\right)}_{P(X\in A)} \underbrace{\left(\sum_{y_j \in \text{Range}(Y)\cap B} P(Y=y_j)\right)}_{P(Y\in B)}$$

Cts case:

$$P(X\in A) = \int_{\cancel{\text{Range}(X)}\cap A} \underbrace{P(X=x)} = \int_A f_X(x)\,dx$$

f_X = PDF of X.

"$\underbrace{P(X=x) = f_X(x)\,dx}$"

$$F_X(x) = \text{CDF of } X$$

$$= \underbrace{P(X < x)}_{P(X \in (-\infty, x))} = \underbrace{\int_{-\infty}^{x} f_X(x)\, dx}_{\int_{(-\infty, x)} f_X(x)\, dx}$$

Intuition for PDF (cts case)

① Replace Σ by $\int$

② Replace $P(X=x)$ by $f_X(x)\, dx$

m

Ind : Wnt $P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$ $\forall$ intervals $A \& B \subseteq \mathbb{R}$.

$$P(X \in A) = \int_A f_X(x)\, dx \quad , \quad P(Y \in B) = \int_B f_Y(y)\, dy$$

$$P(X \in A, Y \in B) = \text{(discrete case)} \sum_{\substack{x_i \in A, \\ y_j \in B}} P(X = x_i, Y = y_j)$$

$$\overset{\text{(cts case)}}{=} \int_{A \times B} f_{X,Y}(x, y)\, dx\, dy$$

$\boxed{\text{Ind}}$: Want

$$\int_{A\times B} f_{X,Y}(x,y)\,dx\,dy = \left(\int_A f_X(x)\,dx\right)\left(\int_B f_Y(y)\,dy\right) \quad (*)$$

$\forall$ intervals A, B.

Note $(*)$ is true $\iff$ $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ $\forall x, y$

(Intuition

$$f_X(x) = P(X=x)\,dx$$
$$f_Y(y) = P(Y=y)\,dy$$
$$f_{X,Y}(x,y) = P(X=x, Y=y)\,dx\,dy$$

"$P(X=x, Y=y)\,dx\,dy = P(X=x)\,dx\;P(Y=y)\,dy$"

Q: $(X, Y) \sim N(\cancel{\underline{\mu}}\ \underline{0}, \underline{\Sigma})$

$\boxed{\text{If } \operatorname{cov}(X,Y) = 0 \text{ are } X \text{ \& } Y \text{ ind?}}$

$$\Sigma = \begin{pmatrix} \operatorname{Var}(X) & \operatorname{Cov}(X,Y) \\ \operatorname{Cov}(X,Y) & \operatorname{Var}(Y) \end{pmatrix}$$

0 (Given)

0 (Given)

Let $\operatorname{Var}(X) = \sigma^2$

$\operatorname{Var}(Y) = \tau^2$

$$\Rightarrow \Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \tau^2 \end{pmatrix}$$

① PDF of (X,Y) $= f_{X,Y}(x,y) = \dfrac{1}{(2\pi\sigma^2\tau^2)}\exp\left(-\dfrac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sigma^2} & 0\\ 0 & \frac{1}{\tau^2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right)$

$Z \sim N(\mu, \Sigma)$ (d dim norm)

PDF of Z $= \dfrac{1}{(2\pi)^{d/2}\det(\Sigma)^{1/2}}\exp\left(-\dfrac{1}{2}(z-\mu)\cdot\Sigma^{-1}(z-\mu)\right)$

$\Rightarrow f_{X,Y}(x,y) = \dfrac{1}{2\pi\sigma^2\tau^2}\exp\left(-\dfrac{x^2}{2\sigma^2}-\dfrac{y^2}{2\tau^2}\right)$

② Compute PDF of $X = f_X(x) = \int_{y \in \mathbb{R}} f_{X,Y}(x,y)\,dy$

① Given $P(X=x_i, Y=y_j)$

Find $P(X=x_i) = \sum_{y_j} P(X=x_i, Y=y_j)$

$$\Rightarrow f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy = \int_{\mathbb{R}} \frac{1}{2\pi\sigma\tau} e^{-\left(\frac{x^2}{2\sigma^2} + \frac{y^2}{2\tau^2}\right)}\,dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\tau^2}} e^{-y^2/2\tau^2}\,dy}_{\text{PDF of a 1D Normal}}$$

$$\Rightarrow \int_{-\infty}^{\infty} = 1$$

$$\boxed{\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}}$$

PDF of a 1D normal.

Similarly $f_Y(y) = \dfrac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{y^2}{2\tau^2}}$

Hence $f_{X,Y}(x,y) = \dfrac{1}{2\pi\sigma\tau} e^{-\frac{x^2}{2\sigma^2} - \frac{y^2}{2\tau^2}} = f_X(x) \cdot f_Y(y)$

$$\boxed{\Rightarrow X \,\&\, Y \text{ are Ind !!}}$$

(Normal Correlation thm)