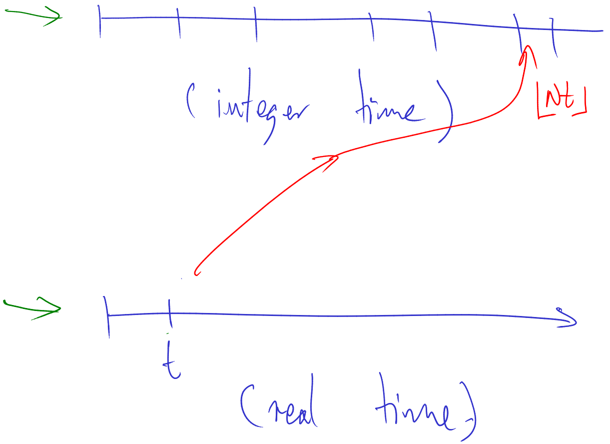
Last time: X_k (iid) $\tilde{E} X_k = 0$, $\tilde{E} X_k^2 = 1$

$$S_n = \sum_{k=1}^{n} X_k$$

$$W^N_n = \frac{1}{\sqrt{N}} S_n$$

Define $W_t = \lim_{N \to \infty} W^N_{\lfloor Nt \rfloor}$



Thm: Above lim exists a.s.

Thm: (1) $t \to W_t$ is cts as a fn of t.

(2) W_s is ind of $W_t - W_s$ & $W_t - W_s \sim N(0, t-s)$

Lemma 8.16. *Let f be a bounded continuous function, fix $T > 0$. By the Markov property we know $\tilde{\boldsymbol{E}}_n f(W^N_{\lfloor NT \rfloor}) = g_n(W^N_n)$ for some function g_n. Set $u(t,x) = \lim_{N\to\infty} g_{\lfloor Nt \rfloor}(x)$. Then $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ and $u(T,x) = f(x)$.*

Last time: (1) $u(T,x) = f(x)$

(2) Wrote recurrence relation for g_n's, use Taylor's thm & got

$(*)$ ... $\dfrac{g_n(x) - g_{n+1}(x)}{1/N} = \frac{1}{2} g_{n+1}''(x) + O\left(\frac{1}{\sqrt{N}}\right)$

(3) $u(t,x) = \lim_{N\to\infty} g_{\lfloor Nt \rfloor}(x) \Rightarrow \partial_x^2 u = \lim_{N\to\infty} g_{\lfloor Nt \rfloor}''(x)$ ... $(**)$

(4) $u(t,x) = \lim\limits_{N\to\infty} g_{\lfloor Nt \rfloor}(x)$ . . .

$$\Rightarrow u(t,x) \approx g_{\lfloor Nt \rfloor}(x) + \text{something small.}$$

$$\left.\begin{aligned} &\Rightarrow u\left(t+\frac{1}{N}, x\right) \approx g_{\lfloor N(t+\frac{1}{N}) \rfloor}(x) + \text{something small} \\ &\qquad\qquad \approx g_{\lfloor Nt \rfloor + 1}(x) + \text{'' small.} \end{aligned}\right\}$$

$$\Rightarrow \frac{u(t,x) - u(t+\frac{1}{N},x)}{1/N} \approx \left(\frac{g_{\lfloor Nt \rfloor}(x) - g_{\lfloor Nt \rfloor + 1}(x)}{1/N} \right) = \underbrace{\frac{1}{2} g''_{\lfloor Nt \rfloor + 1}(x)} + O\left(\frac{1}{\sqrt{N}}\right)$$

(**)

$$\approx \frac{1}{2}\,\partial_x^2 u(t,x)$$

$$\Rightarrow \quad \frac{u(t,x) - u\left(t+\frac{1}{N},x\right)}{1/N} = \frac{1}{2}\,\partial_x^2 u(t,x) + O\left(\frac{1}{\sqrt{N}}\right)$$

$$\boxed{-\partial_t u(t,x) = \frac{1}{2}\,\partial_x u(t,x)}$$

QED.

Lemma 8.17. *Suppose $u = u(t,x)$ satisfies $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ for $t < T$ and $u(T,x) = f(x)$, then*

$$u(t,x) = \int_{\mathbb{R}} f(y) G_{T-t}(x-y)\,dy = \int_{\mathbb{R}} f(x-y) G_{T-t}(y)\,dy\,, \qquad \text{where } G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\,.$$

Density of Normal dist mean 0 & variance t

Pf: Start with the formula for u.

check $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ & $u(T,x) = f$

① $\partial_x G_t(x) = \dfrac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \cdot \left(\dfrac{-x}{t}\right)$

② $\frac{1}{2}\partial_x^2 G_t(x) = \dfrac{1}{2\sqrt{2\pi t}} e^{-x^2/2t} \cdot \left(\dfrac{x^2}{t^3}\right) + \dfrac{1}{2\sqrt{2\pi t}} e^{-x^2/2t} \left(\dfrac{-1}{t}\right)$

③ $\partial_t G = \partial_t \left(\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \right) = -\frac{1}{2\sqrt{2\pi t^3}} \cdot e^{-x^2/2t} + \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \left(\frac{x^2}{2t^2} \right)$

$\Rightarrow$ ④ $\partial_t G_t = \frac{1}{2} \partial_x^2 G_t$.

$\Rightarrow$ ⑤ $\partial_t (G_{T-t}) + \frac{1}{2} \partial_x^2 G_{T-t} = 0$

$\Rightarrow$ ⑥ $u(t,x) = \int_{R} f(y)\, G_{T-t}(x-y)\, dy$

$$\Rightarrow \left(\partial_t + \frac{1}{2}\partial_x^2\right) u(t,x) = \left(\partial_t + \frac{1}{2}\partial_x^2\right) \int_{\mathbb{R}} f(y)\, G_{T-t}(x-y)\, dy$$

$$= \int_{\mathbb{R}} f(y) \underbrace{\left(\partial_t + \frac{1}{2}\partial_x^2\right) \underline{G_{T-t}}(x-y)}_{=0}\, dy$$

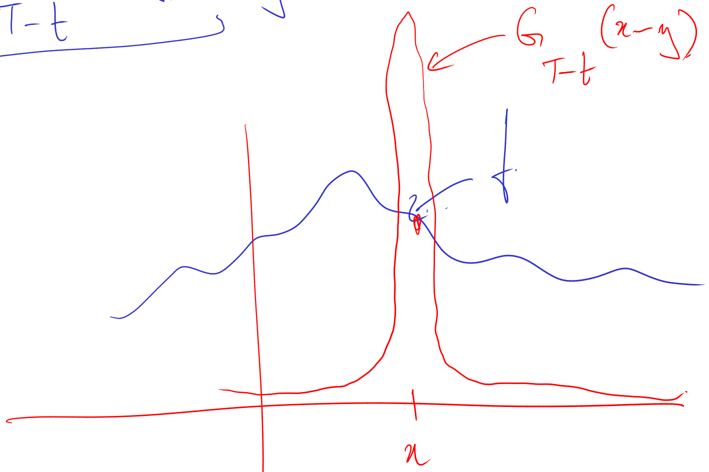
$$\Rightarrow \partial_t u + \frac{1}{2}\partial_x^2 u = 0$$

7 Also NTC $u(x,T) = f(T)$.

$$u(x,t) = \int f(y) \underbrace{G_{T-t}(x-y)} \, dy$$

(t close to T)

$$\xrightarrow[t \to T.]{} f(x)$$



QED

Proof of Proposition 8.14

To $W_T \sim N(0, T)$.

Enough to show $E f(W_T) = \int f(x) \, (\text{PDF of } N(0,T)) \, dx$

$\forall$ bdd cts fns f.

Note: $u(0,0) = E f(W_T)$ where u is the fn from the above lemma.

Also know $u(0,0) = \int_{-\infty}^{\infty} f(y) \, G_{T-0}(0-y) \, dy = \int_{-\infty}^{\infty} f(y) \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} \, dy$

PDF of $N(0,t)$

QED.

Definition 8.18. We say a random variable Y is $\boxed{\mathcal{F}_t \text{ measurable}}$ if $Y = \lim_{n\to\infty} f_n(W_{t_1}, \ldots, W_{t_n})$ where $\boxed{t_i \leqslant t}$ for all i.

Definition 8.19. If $Y = f(W_{t_1}, \ldots, W_{t_n})$ for some function f and $0 \leqslant t_1 \cdots < t_n$, define $\boxed{\boldsymbol{E}_t Y} = \lim_{N\to\infty} \tilde{\boldsymbol{E}}_{\lfloor Nt \rfloor} f(W^N_{\lfloor Nt_1 \rfloor}, \ldots, W^N_{\lfloor Nt_n \rfloor})$

Remark 8.20. $\boldsymbol{E}_t f(W_T) = u(t, W_t)$, where u is the function in Lemma 8.16.

Proposition 8.21. *W is a martingale.*

(don't need $t_i \leq t$)

Disc time mg: $\quad E_m M_n = M_m \quad \forall\, m \leq n.$

cts time $\quad E_s M_t = M_s$