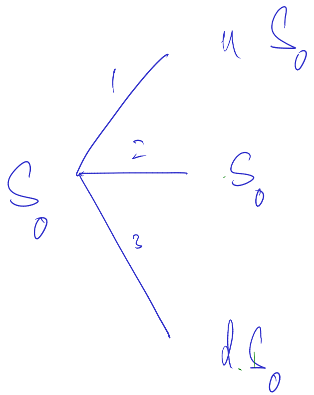


Q: Arb free

$$\boxed{d < 1+r < u.}$$

$$\boxed{d < 1 < u}$$

Add assumption.

Say $1+r \le d < u$

Q $\Rightarrow$ Arb? Yes

Time 0

Buy 1 share of stock

Borrow $S_0$ from bank

Time 1: Worth $S_1 - (1+r)S_0$

$$S_1 - (1+r)S_0 \begin{cases} (u-(1+r))S_0 \ge 0 \\ -rS_0 \quad \leftarrow \text{+ve.} \\ (d-(1+r))S_0 \ge 0 \end{cases}$$

Say $d < 1$

$1+r \le d \le 1$

$\Rightarrow r \le 0$

Find a RNM:

$$\&\begin{cases} (1+r)\cancel{S_0} = u\cancel{S_0}\,\tilde{p}(1) + \cancel{S_0}\,\tilde{p}(2) + d\cancel{S_0}\,\tilde{p}(3) \\ \tilde{p}(1) + \tilde{p}(2) + \tilde{p}(3) = 1 \end{cases} \Rightarrow \text{Can } \underline{\text{NEVER}} \text{ be complete}$$

Lin alg refresher: $m$ linear eq in $\underline{n}$ variables

(1) Homogeneous case: Solve $\underline{\displaystyle\sum_{j=1}^{n} a_{ij} x_j = 0}$ (Given $a_{ij}$ Solve for $x_j$)

$\forall i \in \{1, \dots m\}$.

$$A = \begin{pmatrix} \boxed{\cdots \; a_{ij} \; \cdots} \end{pmatrix} \quad \& \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j$$

i.e. $\displaystyle\sum_{j=1}^{n} a_{ij} x_j = 0 \quad \forall i \in \{1, \dots m\} \iff \underbrace{Ax = 0}$

$\iff \underline{x \in \ker(A)} \quad (x \in \mathrm{Null}(A))$

$$\boxed{\text{Rank Nullity: } \underline{\dim(\ker(A))} + \dim(\mathrm{Range}(A)) = \dim(\mathrm{domain})}$$

Refresher: ① $v_1 \cdots v_n \in \mathbb{R}^d$. $v_i$ is linearly independent if

whenever $\sum \alpha_i v_i = 0 \implies \alpha_i = 0 \ \forall i$

(e.g. $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ )

② $V \subseteq \mathbb{R}^d$ a subspace (i.e. $\forall u, v \in V$, $\alpha u + \beta v \in V \ \forall \alpha, \beta \in \mathbb{R}$)

We say $\{v_1, \cdots v_n\}$ is a basis of $V$ if

ⓐ $\operatorname{span}\{v_1 \cdots v_n\} = \{\sum \alpha_i v_i \mid \alpha_i \in \mathbb{R}\} = V$

② $\{v_1, \cdots v_n\}$ is L.I.

③ $\underline{\dim(V)}$ = # elements in a basis.

$\dim(\text{Plane}) = 2 \qquad \dim(\text{line}) = 1$

$\dim(\mathbb{R}^3) = 3 \qquad (\dim(\mathbb{R}^d) = d).$

Back to rank nullity: $\underbrace{\dim(\ker(A))}_{\text{IDK}} + \underbrace{\dim(\text{Range}(A))}_{\leq m} = \underbrace{\dim(\text{domain})}_{n}.$

$A = \underbrace{\begin{pmatrix} a_{ij} \end{pmatrix}}_{n \text{ colums}} \Big\} m \text{ rows}$

$x \in \mathbb{R}^n$

$Ax \in \underline{\mathbb{R}^m}$

$\Rightarrow \dim(\mathrm{Ker}(A)) \geq n - m$

If $n > m$ (# vars > # eqns)

then $\underbrace{\dim(\mathrm{Ker}(A))}_{\text{\# of L.I. sols to } \boxed{Ax = 0}} > 0$ ($\Rightarrow \exists \infty'ly$ many solutions)

② Non Homg case:

m

$\downarrow$

RN Eqns

$$\&\begin{cases} (1+r)\cancel{S_0} = u\cancel{S_0}\,\tilde{p}(1) + \cancel{S_0}\,\tilde{p}(2) + d\cancel{S_0}\,\tilde{p}(3) \\ \tilde{p}(1) + \tilde{p}(2) + \tilde{p}(3) = 1 \end{cases} \Longrightarrow \text{Can } \underline{\text{NEVER}} \text{ be complete}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} u & 1 & d \\ 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \tilde{p}(1) \\ \tilde{p}(2) \\ \tilde{p}(3) \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1+r \\ 1 \end{pmatrix}}_{V.}$$

(2) Non homg case : Want to solve $\underline{Ax = v}$

Note : If $\underline{\underline{x'}}$ solves $\underline{Ax = v}$ (Guess a "particular sol")

& $y$ solves $Ay = 0$ ||

Then $x = \cancel{x}\, x' + y$ solves $Ax = \underbrace{Ax'}_{v} + \underbrace{Ay}_{0} = v$

$\Rightarrow$ To solve $m$ $\underline{in}$ homg eqns in $n$ variables $(m < n)$

$\longrightarrow$ find a particular sol

& add to it any soln of the hom eqns ←

⇒ If $m < n$ & $\exists\, 1$ soln to the system $Ax = v$ then $\exists\, \infty$ly many

Lets see if we can solve
$$\begin{pmatrix} u & 1 & d \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{p}(1) \\ \tilde{p}(2) \\ \tilde{p}(3) \end{pmatrix} = \begin{pmatrix} 1+r \\ 1 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & \frac{1}{n} & \frac{d}{n} \\ 0 & 1-\frac{1}{n} & 1-\frac{d}{n} \end{pmatrix} \begin{pmatrix} \tilde{p}(1) \\ \tilde{p}(2) \\ \tilde{p}(3) \end{pmatrix} = \begin{pmatrix} \frac{1+r}{n} \\ 1-\frac{1+r}{n} \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & \frac{1}{n} & \frac{d}{n} \\ & 1 & \frac{\cancel{n}}{n-1}\left(\frac{n-d}{\cancel{n}}\right) \end{pmatrix} \begin{pmatrix} \tilde{p} \end{pmatrix} = \begin{pmatrix} \frac{1+r}{n} \\ \frac{\cancel{n}}{n-1}\left(\frac{n-(1+r)}{\cancel{n}}\right) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{n}\left(d - \frac{(u-d)}{n-1}\right) \\ 0 & 1 & \frac{u-d}{n-1} \end{pmatrix} \tilde{p} = \begin{pmatrix} \left(1+r - \frac{u(1+r)}{n-1}\right)\frac{1}{n} \\ \frac{u-(1+r)}{n-1} \end{pmatrix}$$

Choose $\underline{\tilde{p}(3)} = \lambda$ :

$$\Rightarrow \tilde{p}_1 = \left(1+r - \frac{u(1+r)}{n-1}\right)\frac{1}{n} - \frac{1}{n}\left(d - \frac{(u-d)}{n-1}\right)\lambda$$

$$\& \quad \tilde{p}_2 = \left(\frac{u-(1+r)}{n-1}\right) - \left(\frac{u-d}{n-1}\right)\lambda$$

Just check that one $\underline{\lambda > 0}$ gives $\tilde{p}_2 > 0$ & $\tilde{p}_1 > 0$

(by cont $\Rightarrow \exists\ \infty$ ly many $R \sim M$)