Last time: $X_n \longrightarrow$ iid., $E X_n = \mu$, $Var(X_n) = \sigma^2$.

LLN: $S_n = \sum_1^n X_k$, $\frac{S_n}{n} \xrightarrow{n \to \infty} \mu$.

LLN: $\frac{(S_n - n\mu)}{n} \xrightarrow[a.s.]{n \to \infty} 0$ (LLN).

CLT: $\frac{(S_n - n\mu)}{\sqrt{n}} \xrightarrow{n \to \infty} N(0, \sigma^2)$

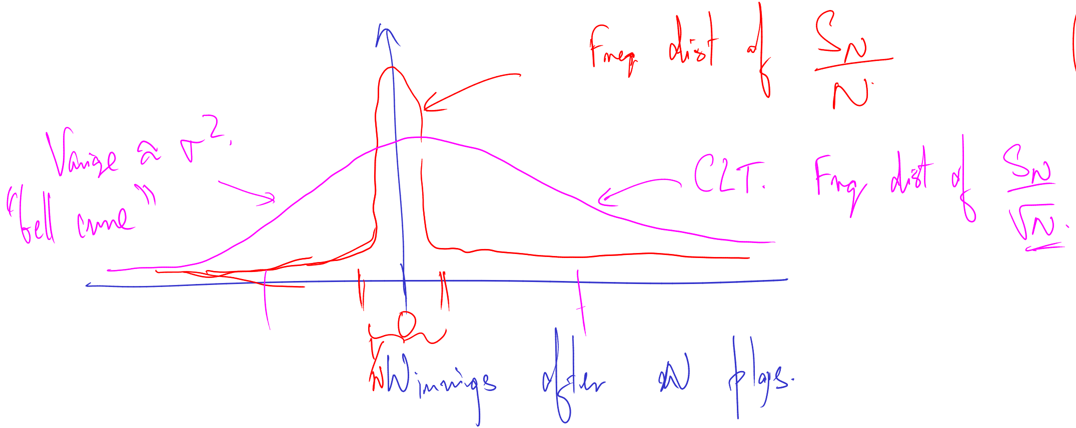
weak conv.

($\forall$ bdd cts f, $\lim_{n \to \infty} E f\left(\frac{S_n - n\mu}{\sqrt{n}}\right) = E f(N(0, \sigma^2))$)

$$= \int_{-\infty}^{\infty} f(x)\, G_{\sigma^2}(x)\, dx$$

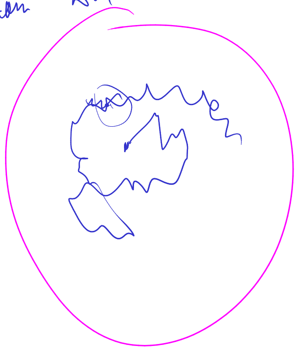
Where $\underline{G_{\sigma^2}(x)} = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}\, dx$

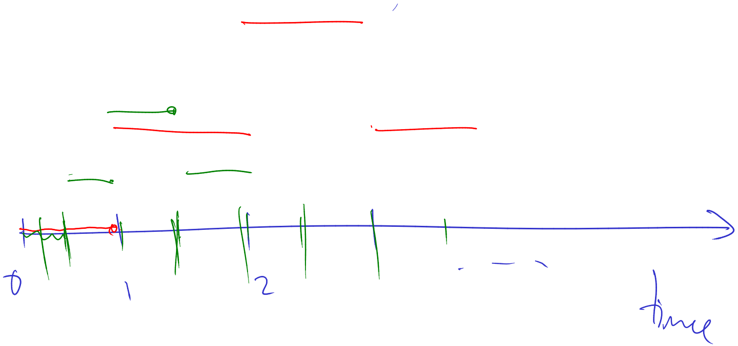
(Say $\mu = 0$).



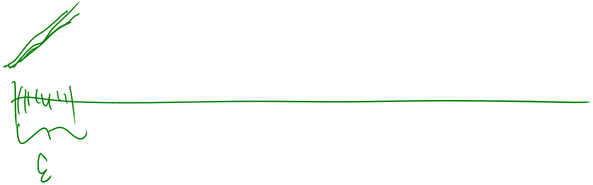
Brownian motion

"Cts time random walk"





Discrete RW.



8.3. **Brownian motion.**

- Suppose now $X_1, X_2, \dots$ are i.i.d. $\mathbb{R}$ valued random variables.
- Use $\tilde{\boldsymbol{P}}$ to denote the probability measure, and $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ to denote the associated expectation / conditional expectation.
- Assume $\tilde{\boldsymbol{E}} X_n = 0$, and $\tilde{\boldsymbol{E}} X_n^2 = 1$.

$\left(S_n = \sum_1^n X_k \right)$

Theorem 8.10. *Let* $W_n^N = \frac{1}{\sqrt{N}} S_n = \frac{1}{\sqrt{N}} \sum_1^n X_k$. *Then* $\lim_{N \to \infty} W_{\lfloor Nt \rfloor}^N$ *exists almost surely.*

(Subscript $\to$ time)

Theorem 8.11. *(1) The function* $t \mapsto W_t$ *is continuous almost surely, and* $W_0 = 0$.
(2) If $0 = t_0 < t_1 < \cdots t_n$, *then* $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, $\dots$, $W_{t_n} - W_{t_{n-1}}$ *are independent and* $\boxed{W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})}$.

Remark 8.12. Typically one changes the probability space to ensure the function $t \mapsto W_t$ is continuous *surely*.

Definition 8.13. The process W above is called a standard (one dimensional) Brownian motion.

Let $W_t = \lim_{N \to \infty} W_{\lfloor Nt \rfloor}^N$

$\lfloor Nt \rfloor$ = largest int $\leq Nt$
$= \text{floor}(Nt)$

$W_t - W_s \sim N(0, t - s)$

The full proof of Theorems 8.10 and 8.11 are technical and beyond the scope of this course. However, we can prove a weaker result here:

Proposition 8.14. $W_T \sim \mathcal{N}(0, T)$.

Remark 8.15. The above is simply the central limit theorem (which we never proved). We will prove it here. Our proof can also be modified to prove that W has independent normally distributed increments.

Lemma 8.16. *Let f be a bounded continuous function, fix $T > 0$. By the Markov property we know $\tilde{\boldsymbol{E}}_n f(W^N_{\lfloor NT \rfloor}) = g_n(W^N_n)$ for some function g_n. Set $u(t, x) = \lim_{N\to\infty} g_{\lfloor Nt \rfloor}(x)$. Then $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ and $u(T, x) = f(x)$.*

Markov.

$$W^N_{\lfloor NT \rfloor} = \frac{1}{\sqrt{N}} \sum_{k=0}^{\lfloor NT \rfloor} X_k \leftarrow \text{Markov Process}$$

Pf: Note: Clearly $u(T, x) = \lim_{N\to\infty} g_{\lfloor NT \rfloor}(x) = f(x)$

TOU: $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$

① Find a recurrence relation for g_n's

$$g_n(W_n^N) = \tilde{E}_n f\left(W^N_{\lfloor NT \rfloor}\right)$$

$$W_n^N = \frac{1}{\sqrt{N}} \sum_1^n X_k$$

$$= \tilde{E}_n \tilde{E}_{n+1} f\left(W^N_{\lfloor NT \rfloor}\right)$$

$$= \tilde{E}_n \, g_{n+1}\left(W^N_{n+1}\right) = \tilde{E}_n \underbrace{g_{n+1}\left(W_n^N + \frac{1}{\sqrt{N}} X_{n+1}\right)}$$

Note $W_n^N \longrightarrow \mathcal{F}_n$ meas

$\frac{1}{\sqrt{N}} X_{n+1} \longrightarrow$ indep of $\mathcal{F}_n$ $\Big\}$ — Indep lemma

& Say Range of $X_{n+1} = \{x_1, \dots x_m\}$ & $p_i = P(X_{n+1} = x_i)$.

Then $g_n(\underbrace{W_n^N}_{x}) = \tilde{E}_n\left(g_{n+1}\left(W_n^N + \frac{1}{\sqrt{N}} X_{n+1}\right)\right)$

$$\overset{\text{indep lemma}}{=} \sum_{i=1}^{M} p_i \, g_{n+1}\left(\underbrace{W_n^N}_{x} + \frac{x_i}{\sqrt{N}}\right)$$

Let $x = W_n^N$.

$$\boxed{\Rightarrow \quad g_n(x) = \sum_{i=1}^{m} g_{n+1}\left(x + \frac{x_i}{\sqrt{N}}\right) p_i}$$

Taylor expand g_{n+1}: $\quad g_{n+1}(x+h) = g(x) + h g'(x) + \frac{1}{2} h^2 g''(x) + O(h^3)$

$$\hookrightarrow g_n(x) = \sum_{i=1}^{m} p_i \left[g_{n+1}(x) + \frac{x_i}{\sqrt{N}} g'(x) + \frac{1}{2N} x_i^2 g''(x) \right] + O\left(\frac{1}{N^{3/2}}\right)$$

$$= g_{n+1}(x) + \frac{1}{\sqrt{N}} \underbrace{\left(\sum p_i x_i\right)}_{EX_{n+1}=0} g'(x) + \frac{1}{2N} \underbrace{\left(\sum x_i^2 p_i\right)}_{EX_{n+1}^2=1} g''(x).$$

$$\boxed{\Rightarrow \underbrace{\frac{g_n(x)-g_{n+1}(x)}{1/N}}_{\substack{\downarrow\ \text{(Next time)} \\ -\partial_t u(x)}} = 0 + \underbrace{\frac{1}{2}\underline{\underline{g''_{n+1}(x)}}}_{\substack{\downarrow\ \text{(Next time)} \\ \frac{1}{2}\partial_x^2 u(x)}} + \underbrace{O\left(\frac{1}{\sqrt{N}}\right)}_{\substack{\downarrow \\ 0}}}$$