(4) (1) Law of Large #'s.

$X_1, X_2, X_3, \ldots$ (iid).

$\underline{E} X_i = \mu$

Say we play N times

- Avg winnings: $\frac{1}{N} \sum_{k=1}^{N} X_k$?
- Expected winnings $= \mu$

$\frac{1}{N} \sum_{k=1}^{N} X_k \to \mu$ as $N \to \infty$

8.1. **Law of large numbers.** Now consider infinitely many i.i.d. random variables $X_1, X_2, \dots$.

Theorem 8.1 (Weak law of large numbers). *Suppose* $\boldsymbol{E} X_n = \mu$ *and* $\operatorname{Var} X_n = \sigma^2 < \infty$*, and let* $S_n = \sum_1^n X_k$. *Then* $\operatorname{Var}(S_n/n) \to 0$*, and hence for any* $\varepsilon > 0$, $\lim_{n\to\infty} \boldsymbol{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = 0$.

total winnings after n plays.

Lemma 8.2 (Chebychev's inequality). *For any* $\varepsilon > 0$, $\boldsymbol{P}(X > \varepsilon) \leqslant \frac{1}{\varepsilon} \boldsymbol{E}|X|$.

Pf: Note $\varepsilon \mathbb{1}_{\{X > \varepsilon\}} \leq |X|$

$\Rightarrow \varepsilon E \mathbb{1}_{\{X>\varepsilon\}} \leq E|X|$

$\Rightarrow P(X > \varepsilon) \leq \frac{1}{\varepsilon} E|X|$. QED.

Proof of Theorem 8.1 $EX_n = \mu$, $E(X_n-\mu)^2 = \sigma^2$. X_n's are i.i.d.

$$S_n = \sum_{1}^{n} X_k$$

① NTS $\mathrm{Var}\left(\frac{S_n}{n}\right) \longrightarrow 0$.

Pf: Note $\mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(S_n) = \frac{1}{n^2}\sum_{1}^{n}\mathrm{Var}(X_k) = \frac{\sigma^2}{n}$.

$$\Rightarrow \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0$$

(∵ X & Y are indep $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$)

② NTS $\forall \varepsilon > 0$, $P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \xrightarrow{n\to\infty} 0$

Pf: $P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = P\left(\left(\frac{S_n}{n} - \mu\right)^2 > \varepsilon^2\right)$

$$\overset{\text{Chebyshev}}{\leq} \frac{1}{\varepsilon^2} \underbrace{E\left(\frac{S_n}{n} - \mu\right)^2}_{= \operatorname{Var}\left(\frac{S_n}{n}\right)}$$

$$= \frac{1}{\varepsilon^2} \cdot \frac{\sigma^2}{n} \longrightarrow 0$$

QED.

Theorem 8.3 (Strong law of large numbers)**.** *Under the same assumptions as Theorem 8.1,* $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ *almost surely.*

Expected return

avg winnings after n plays

(Pf is harder)

Play a fair slot machine N times.

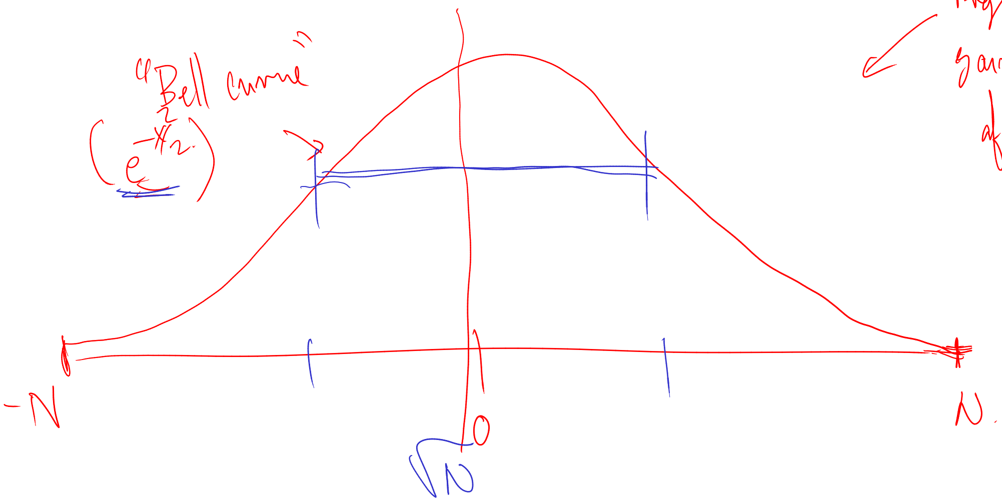
$X_k \to$ outcome of kth slot. (iid)

$X_k = 1$ with prob $1/2$

$X_k = -1$ with prob $1/2$

$E X_k = 0$

"Bell curve" $\left(e^{-x^2/2}\right)$



Freq dist of your winnings after N plays.

8.2. Central limit theorem.

Theorem 8.4. *Let X_n be a sequence of $\mathbb{R}^d$ valued, i.i.d. random variables be such that $\boldsymbol{E}X_n^i = \mu_i$ and $\operatorname{cov}(X_n^i, X_n^j) = \Sigma_{i,j}$. Let $S_N = \sum_1^N X_n$. Then $(S_N - N\mu)/\sqrt{N}$ converges weakly to $\mathcal{N}(\mu, \Sigma)$.*

Definition 8.5. We say a sequence of random variables Y_n converges weakly to a random variable Z if $\boldsymbol{E}f(Y_n) \to \boldsymbol{E}f(Z)$ for every bounded continuous function f.

Definition 8.6. Let $\mu \in \mathbb{R}^d$, and Σ be a $d \times d$ *covariance matrix* (positive semi-definite, symmetric).

(1) $\mathcal{N}(\mu, \Sigma)$ denotes a *normally distributed random variable* with mean μ and covariance matrix Σ.

(2) When Σ is invertible, the probability density function of $\mathcal{N}(\mu, \Sigma)$ is $\dfrac{1}{(2\pi \det(\Sigma))^{d/2}} \exp\Big(-\dfrac{1}{2}(x-\mu)\cdot\Sigma^{-1}(x-\mu)\Big)$

(3) When $d = 1$, $\Sigma = \sigma^2$ the PDF of $\mathcal{N}(\mu, \sigma^2)$ is $\dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)/(2\sigma^2)}$.

(4) When $\mu = 0$, $\sigma = 1$, $\mathcal{N}(0, 1)$ is called the *standard normal*, and its PDF is the *Gaussian* $G(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Definition 8.7. We say p is the probability density function (PDF) of a d-dimensional random variable X if $\boldsymbol{P}(X \in A) = \int_A p(x)\,dx$ for all cubes $A \subseteq \mathbb{R}^d$.

Remark 8.8. Equivalently, p is the PDF of X if $\boldsymbol{E}f(X) = \int_{\mathbb{R}^d} f(x)p(x)\,dx$ for every bounded continuous function f.

Remark 8.9. We will prove Theorem 8.4 during the course of the construction of Brownian motion.

S.LLL: $EX_n = 0$. $\qquad \frac{S_n}{n} \longrightarrow \mu = 0$ a.s.

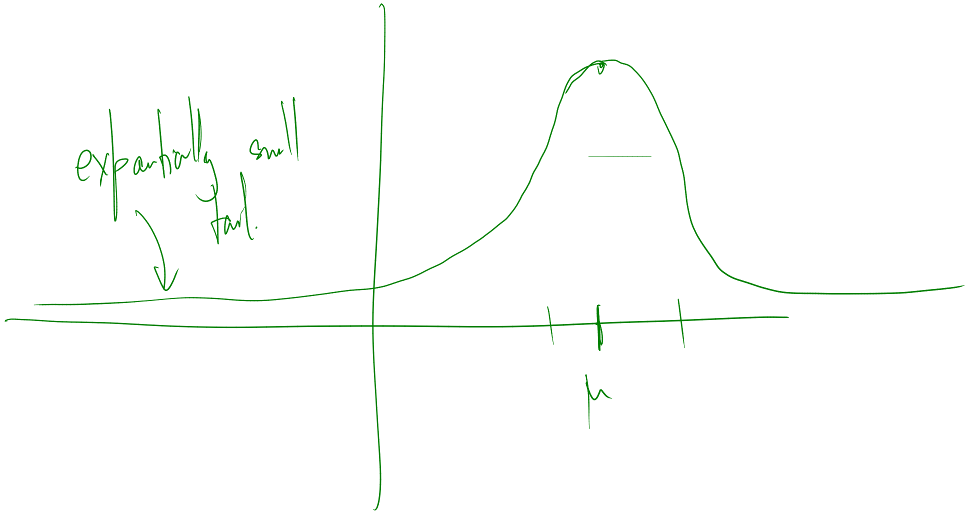
CLT: $EX_n = 0$: $\qquad \frac{S_n}{\sqrt{n}} \xrightarrow{\text{"weakly"}} N(0, \Sigma)$

Normally dist RV mean 0, cov. Σ

$Y_n \to Z$ weakly if $Ef(Y_n) \longrightarrow Ef(Z)$ $\forall$ bdd cts fns f.

$\iff$ for cts RVs, $\lim_{n\to\infty} P(Y_n \in A) = P(Z \in A)$ ($\forall A \subseteq \mathbb{R}$ intervals)

$$G_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ (Gaussian)



√

X is a $\mathbb{R}$ valued RV.

f is the PDF of X if $\forall\, a, b \in \mathbb{R}$, $P(X \in (a,b)) = \int_a^b f(x)\, dx$

$\iff \forall$ bdd cts fn f, $\quad E f(X) = \int_{\mathbb{R}} f(x)\, p(x)\, dx$

(Note: For disc RV's, p is the PMF.

Y is a RV. $\quad E f(Y) = \sum f(y_i)\, P(Y = y_i)$

(Range $Y = \{y_1, \dots, y_n\}$))

8.3. **Brownian motion.**

- Suppose now $X_1, X_2, \dots$ are i.i.d. $\mathbb{R}$ valued random variables.
- Use $\tilde{\boldsymbol{P}}$ to denote the probability measure, and $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ to denote the associated expectation / conditional expectation.
- Assume $\tilde{\boldsymbol{E}} X_n = 0$, and $\tilde{\boldsymbol{E}} X_n^2 = 1$.

Theorem 8.10. *Let* $W_n^N = \frac{1}{\sqrt{N}} S_n = \frac{1}{\sqrt{N}} \sum_1^n X_k$. *Then* $\lim_{N \to \infty} W_{\lfloor Nt \rfloor}^N$ *exists almost surely.*

Theorem 8.11. *(1) The function* $t \mapsto W_t$ *is continuous almost surely, and* $W_0 = 0$.
(2) If $0 = t_0 < t_1 < \cdots t_n$, *then* $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, $\dots$, $W_{t_n} - W_{t_{n-1}}$ *are independent and* $W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$.

Remark 8.12. Typically one changes the probability space to ensure the function $t \mapsto W_t$ is continuous *surely*.

Definition 8.13. The process W above is called a standard (one dimensional) Brownian motion.