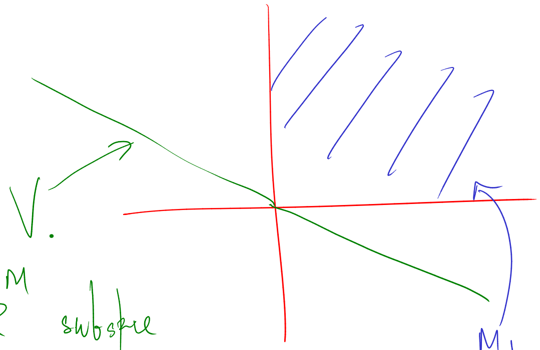
HW 12 Q2

NTS: V has a unique ~~norm~~ unit normal in $\mathring{Q}$

$\iff V \cap \bar{Q} = \{0\}$



$V \subseteq \mathbb{R}^M$ subspace

$\boxed{\text{Given } \dim(V) = M-1}$

(Typo in problem statement will be fixed shortly)

$\bar{Q} = \{v \in \mathbb{R}^M \mid v_i \geq 0\}$

$\mathring{Q} = \{v \in \mathbb{R}^M \mid v_i > 0\}$

Forward: Assume $\exists! \, \hat{n} \in \overset{\circ}{Q}$, $|\hat{n}| = 1$. $\hat{n} \perp V$ (i.e. $\hat{n} \cdot v = 0$ $\forall v \in V$)

NTS $V \cap \bar{Q} = \{0\}$.

Say $\exists \, v \in V \cap \bar{Q}$, $v \neq 0$ $(v = (v_1, v_2, \cdots v_M))$

$\Rightarrow v_i \geq 0$

$\rightarrow$ Note $\underbrace{\hat{n} \cdot v = \sum \hat{n}_i v_i > 0}$

(at least one $v_i > 0 \Rightarrow \hat{n}_i v_i > 0$
& all other $\hat{n}_j v_j \geq 0$) $\Rightarrow V \cap \bar{Q} = \{0\}$

~~Rank nullity~~:

Let $V^\perp = \{ w \in \mathbb{R}^M \mid w \cdot v = 0 \ \forall v \in V \}$

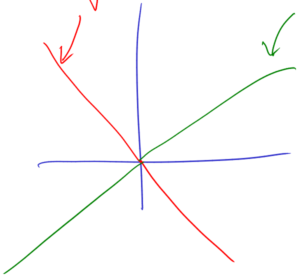
~~R.N~~ $\Rightarrow$ $\dim(V^\perp) + \boxed{\dim(V)} = M$.

$\boxed{\text{Given} = M-1}$

$\dim(V) = 1$ $\quad M = 2$

V

$V^\perp$ $\quad \dim(V^\perp) = 1$



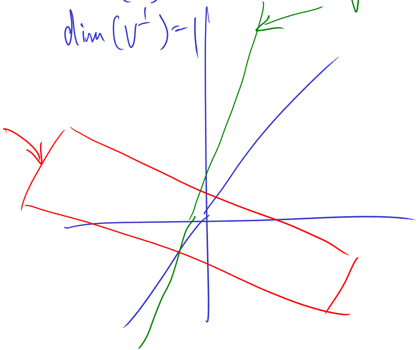
$M = 3$

$\dim(V) = 2$

$\dim(V^\perp) = 1$

$V^\perp$

V



Reverse direction: Given $\dim(V) = M-1$ & $\boxed{V \cap \bar{Q} = \{0\}}$.

NTS $\exists!\ \hat{n} \in \mathring{Q}$

① Know $\dim(V^{\perp}) = 1 \Rightarrow \exists\ \hat{n} \in \mathbb{R}^M$ + $\underline{|\hat{n}| = 1}$ & $\hat{n} \perp V$

② WL assume some coordinate $\hat{n}_i > 0$

③ Say for simplicity $\hat{n}_1 > 0$

④ Claim: $\hat{n}_2 > 0$ (needs Pf).

Note : $V = \{ v \in \mathbb{R}^M \mid v \cdot \hat{n} = 0 \}$ $\quad (\because \dim(V) = M-1)$

$$\hat{n} = \begin{pmatrix} \sim & (>0) \\ \sim & (\text{say} < 0) \\ * & \\ * & \\ * & \end{pmatrix}$$

Know $V = (\hat{n})^{\perp}$ & $V \cap \bar{Q} = \{0\}$

Assume, for contradiction $\hat{n}_2 < 0$

$$\hat{n} = \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \\ * \\ * \\ * \end{pmatrix}$$

$\hat{n}_1 > 0$

$\hat{n}_2 \leq 0$ (assume f.c).

Choose $v = \begin{pmatrix} -\hat{n}_2 \\ \hat{n}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \boxed{v \cdot \hat{n} = 0} \Rightarrow v \in V$

& $v \in \bar{Q}$ & $v \neq 0$

5 Same thing $\forall$ other coordinates..

QED.

m

$\boxed{\text{Q3a}}$ $\Omega = \{1, \dots, M\}^{\mathbb{N}}$, fix $\omega' = (\omega_1, \dots, \omega_n)$

$$\mathcal{U} = \left\{ \begin{pmatrix} \Delta_n^{(\omega')} \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n^{(\omega')} \cdot S_{n+1}(\omega', M) \end{pmatrix} \;\middle|\; \Delta_n(\omega') \in \mathbb{R}^{d+1} \right\}$$

$$\underline{V} = \left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \;\middle|\; \underbrace{\Delta_n(\omega') \cdot S_n(\omega')}_{\text{Wealth at time } n.} = 0 \right\}.$$

3a] $\dim(V) \le d$.

Note $U = \mathrm{span}\left\{ \underbrace{\begin{pmatrix} S^0_{n+1}(\omega', 1) \\ \vdots \\ S^0_{n+1}(\omega', M) \end{pmatrix}}_{v_0}, \dots, \underbrace{\begin{pmatrix} S^d(\omega', 1) \\ \vdots \\ S^d(\omega', M) \end{pmatrix}}_{v_d} \right\}$

$\Rightarrow U = \mathrm{span}\{v_0, \dots, v_d\} \Rightarrow \dim(U) \le \underline{d+1}$

Note $V = \left\{ \sum \alpha_i v_i \;\middle|\; \underline{\sum \alpha_i S^i_n(\omega') = 0} \right\} \subseteq \mathrm{span}\{v_0, \dots, v_d\} = U$

Note $S_n^0(\omega') > 0$

$\circledast$ $\textcircled{x}$

Claim: $V = \text{span}\{\underbrace{-S_n^1(\omega')v_0 + v_1}, -S_n^2(\omega')v_0 + v_2, \ldots, -S_n^d(\omega') + v_d\}$

$$\{\alpha_0 v_0 + \alpha_1 v_1 \mid \alpha_0 \underbrace{S_n^0(\omega')}_{\neq 0} + \alpha_1 S_n^1(\omega') = 0\}$$

$$= \{\alpha_0 v_0 + \alpha_1 v_1 \mid \alpha_0 = -\frac{\alpha_1 S_n^1(\omega')}{S_n^0(\omega')}\}$$

Note: ① NTS $V = \text{span}\{(*)\}$ as claimed. (Please check)

② Given the claim, $\Rightarrow \dim(V) \leq d$.