Last time:

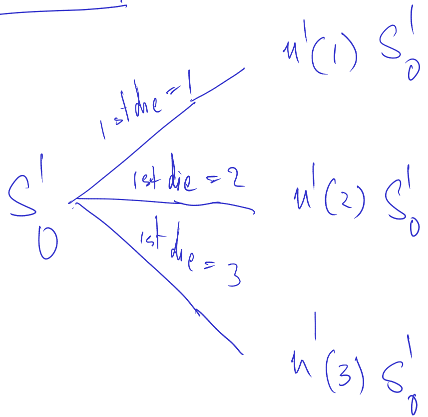
2 indep coins tossed at each time step.

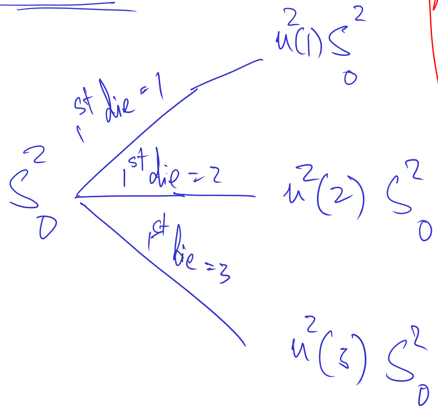
→ conditions for this market to be arb free.

→ <u>Never</u> complete!

Question 7.18. *Consider now repeated rolls of a 3-sided die and for $i \in \{1, 2\}$, let $Z_n^i = u^i(j)$ if the n-th die rolls j. Suppose $S_{n+1}^i = S_n^i Z_{n+1}^i$. Find conditions when this market is complete and arbitrage free.*

And a bank with interest rate $r > -1$





First analyze case $N = 1$ (1 period case)

FTAP 2 $\exists!$ Unique RNM $\iff$ complete & arb free.

Solve for the RNM (time 1)

Want $\tilde{p}(1)$, $\tilde{p}(2)$, $\tilde{p}(3)$ $\ni$ ① $\tilde{E}\left(\frac{S_1^1}{1+r}\right) = S_0^1$

& ② $\tilde{E}\left(\frac{S_1^2}{1+r}\right) = S_0^2$

$\boxed{\tilde{p}(i) = \text{RN prob that the first die rolls } i.}$

① $\Rightarrow$ $\sum_{i=1}^{3} \tilde{p}(i)\, S_1^1(i) = (1+r)\, S_0^1$

② $\Rightarrow$ $\sum_{i=1}^{3} \tilde{p}(i)\, S_1^2(i) = (1+r)\, S_0^2$

& ③ Want $\sum_{i=1}^{3} \tilde{p}(i) = 1$

$\boxed{(\text{\& ④ } \tilde{p}(i) > 0 \ \forall i)}$

$(1)\,\&\,(2) \iff (1+r)\cancel{S_0^j} = \sum_{i=1}^{3} \tilde{p}(i)\, S_1^j(i) = \sum_{i=1}^{3} \tilde{p}(i)\, u^j(i)\cancel{S_0^j} \quad (\text{for } j = 1 \,\&\, 2)$

$\Rightarrow$ Equations are

$$\tilde{p}(1) + \tilde{p}(2) + \tilde{p}(3) = 1 \quad \cdots \quad (1)$$

$$\tilde{p}(1)\, u^1(1) + \tilde{p}(2)\, u^1(2) + \tilde{p}(3)\, u^1(3) = 1+r \quad \cdots \quad (2)$$

$$\tilde{p}(1)\, u^2(1) + \tilde{p}(2)\, u^2(2) + \tilde{p}(3)\, u^2(3) = 1+r \quad \cdots \quad (3)$$

$$\iff \left(\begin{array}{ccc} 1 & 1 & 1 \\ \multicolumn{3}{c}{\longleftarrow u^1 \longrightarrow} \\ \multicolumn{3}{c}{\longleftarrow u^2 \longrightarrow} \end{array}\right) \begin{pmatrix} \tilde{p}(1) \\ \tilde{p}(2) \\ \tilde{p}(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 1+r \\ 1+r \end{pmatrix}$$

Let $A = \begin{pmatrix} | & | & | \\ \longleftarrow & u^1 & \longrightarrow \\ \longleftarrow & u^2 & \longrightarrow \end{pmatrix}$

Q: When does $A \begin{pmatrix} \uparrow \\ \tilde{p} \\ \downarrow \end{pmatrix} = \begin{pmatrix} | \\ 1+r \\ 1+r \end{pmatrix}$ have a unique sol?

$\det(A) \neq 0 \Rightarrow$ unique sol!

Also need to ~~m~~ ensure $\tilde{p}(i) > 0 \quad \forall i$

Given u^j, Can explicitly compute $A^{-1}\begin{pmatrix} | \\ 1+r \\ 1+r \end{pmatrix}$ & check $\tilde{p}(i) > 0 \ \forall i$ (one period)

$\Rightarrow$ Complete & arb free

② More than one period:

Note $$S^{i}_{n+1} = S^{i}_{n}\, \underbrace{Z^{i}_{n+1}}$$ only depends on the $(n+1)^{th}$ die roll.

Choose the RNM with iid die rolls.

Want $$\tilde{E}_n\left(\frac{S^{i}_{n+1}}{1+r}\right) = S^{i}_{n}$$

Note: $$\tilde{E}_n\left(S^{i}_{n} Z^{i}_{n+1}\right) = S^{i}_{n}\, \tilde{E}_n\left(Z^{i}_{n+1}\right)$$

$\Rightarrow$ Need $\tilde{E}_n\left(Z^i_{n+1}\right) = 1+r \quad \forall i$

iid die rolls (RNM) $\Leftrightarrow$ $\underline{\tilde{E}\, Z^i_{n+1}} = 1+r$

Let $\tilde{p}_{n+1}(j)$ = RN prob that the $(n+1)^{th}$ die rolls j.

$$\Rightarrow \tilde{E}\, Z^i_{n+1} = \sum_{j=1}^{3} \tilde{p}_{n+1}(j)\, u^i(j)$$

Same eqn's as before & solve as before!

8. **Black-Scholes Formula**

(1) Suppose now we can trade *continuously in time*.

(2) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.

$C(t) = C(0)e^{rt}$

(3) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$).

(4) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.

(5) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).

(6) The price at time t of a European call with maturity T and strike K is given by

x = spot price of stock

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

$(\tau = T - t)$

$$\text{where} \quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,.$$

(7) We will derive this as the limit of the $\boxed{\text{Binomial model as } N \to \infty}$.

N = CDF of the Normal dist.