HW Q3: $G = (G_0, \ldots G_N)$, $V_N = G_N$, $V_n = \max\{G_n, E_n G_{n+1}\}$

$$\tau_k^* = \min\{n \geq k \mid V_n = G_n\}$$

Want: $\forall \tau_k \geq k$ a.s. NTS $E_k G_{\tau_k^*} \geq E_k G_{\tau_k}$ a.s.

(i.e. τ_k^* solves the optimal stopping problem for G provided you are only allowed to stop <u>after</u> time k.)

Pf when $k=0$: $V = \overset{M}{M} - \underline{A}$ (Doob decomp, A pred inc, $A_0 = 0$)

Step ① Show $A_{\tau_0^*} = \cancel{0}\ A_0$

② $\Rightarrow$ $M_{\tau_0^*} = V_{\tau_0^*} = G_{\tau_0^*}$ (always know $\boxed{M \geq V \geq G}$)

③

Want $EG_{\tau_0^*} \geq EG_\tau$

Know $EG_{\tau_0^*} = EM_{\tau_0^*} \overset{OST}{=} EM_\tau \geq EG_\tau$ ✓

Do the same for $k > 0$.

Step 1 $A_{\tau_k^*} = 0$ ⟵ Can not hope for this to be true for $k>0$

Instead will show : $\boxed{A_{\tau_k^*} = A_k}$

Step 2 $M_{\tau_k^*} = V_{\tau_k^*} = G_{\tau_k^*}$

(Can't hope for this to be true for $k > 0$)

k

$E V_n = E M_n - \boxed{E A_n}$

Instead will show $\boxed{G_{\tau_k^*} = V_{\tau_k^*} = M_{\tau_k^*} - A_k}$ extra.

Snell $\qquad V = M - A \qquad G_n\ V \geq G$

$$A_{\tau_k^*} = A_k \qquad \left(V_{\tau_k^*} = G_{\tau_k^*} \right)$$

$$V_{\tau_k^*} = M_{\tau_k^*} - A_{\tau_k^*} = \underbrace{M_{\tau_k^*} - A_k}_{G_{\tau^*}}$$

Hope this is the right guess

$$\boxed{G_{\tau_k^*} = V_{\tau_k^*} = M_{\tau_k^*} - \underline{A_k}}$$

Step ②: Say $\tau \geq k$ a.s. $\qquad$ Want $E_k G_{\tau_k^*} \geq E_k G_\tau$ a.s.

$$E_k G_{\sigma_k^*} = E_k M_{\sigma_k^*} - A_k$$

$$\overset{OST}{=} M_{\cancel{\sigma_k^*} \wedge k} - A_k$$

$$\overset{OST}{=} E_k M_\tau - A_k \qquad (\because \tau \wedge k = k)$$

$$= E_k (V_\tau + A_\tau) - A_k$$

$$\geq E_k G_\tau + \underbrace{E_k (A_\tau - A_k)}_{\geq 0} \quad (\because \tau \geq k \ \& \ A \text{ inc.})$$

$$\geq E_k G_\tau \quad \checkmark$$

Only NTS $A_{\tau_k^*} = A_k$ a.s. (same proof as the pf in class for $A_{\tau_0^*} = 0$).

Pf when $k = 0$:

$$V_n = \max\{G_n, E_n V_{n+1}\}$$

$$\tau_0^* = \min\{n \mid V_n = G_n\}.$$

$$\boxed{\text{Before } \tau_0^*} \qquad \mathbb{1}_{\{n<\tau_0^*\}} V_n = E_n V_{n+1} \mathbb{1}_{\{n<\tau_0^*\}}$$

Always $\quad M_n = E_n M_{n+1}$

$$\boxed{\text{Before } \tau_0^*} \qquad \& \qquad V = M - A$$

$$\mathbb{1}_{\{n<\tau_0^*\}} V_n = \mathbb{1}_{\{n<\tau_0^*\}} E_n \left(V_{n+1}\right) = \mathbb{1}_{\{n<\tau_0^*\}} E_n \left(M_{n+1} - A_{n+1}\right) = \mathbb{1}_{\{n<\tau_0^*\}} \left(\underbrace{M_n - A_n}_{V_n} + \underline{A_n} - \underline{A_{n+1}}\right)$$

$$\Rightarrow \mathbb{1}_{\{n < \tau_0^*\}} (A_{n+1} - A_n) = 0 \Rightarrow A_{\tau_0^*} = 0$$

$k > 0$. $$\mathbb{1}_{\{n < \tau_k^*\}} (A_{n+1} - A_n) = 0 \Rightarrow \underline{A_{\tau_k^*}} = \underline{\underline{A_k}}$$