7.4. **Examples and Consequences.**

Proposition 7.14. *Suppose the market model Section 7.1 is complete and arbitrage free, and let* $\tilde{\boldsymbol{P}}$ *be the unique risk neutral measure. If* $D_n X_n$ *is a* $\tilde{\boldsymbol{P}}$ *martingale, then* X_n *must be the wealth of a self financing portfolio.*

Remark 7.15. We've already seen in Lemma 7.5 that if a (not necessarily unique) risk neutral measure exists, then the discounted wealth of any self financing portfolio must be a martingale under it.

Remark 7.16. All pricing results/formulae we derived for the Binomial model that only relied on the analog of Proposition 7.14 will hold in complete arbitrage free markets.

Pf: Assume $D_n X_n$ is a $\tilde{P}$ mg (i.e. $\tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n$).

Reminder: ① $D_n = \frac{1}{S^0_n}$ ($S^0_n \to$ Bank process)

② Say $\Delta_n \to$ trading strat. (adapted)

$\Delta_n = (\Delta^0_n, \Delta^1_n, \cdots \Delta^d_n)$

Wealth at time n $= \Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$

$$\boxed{\text{Self financing : } \Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$$

Assume $D_n X_n$ is a $\tilde{P}$ mg.

Need to find Δ_n s.t. $X_n = \Delta_n \cdot S_n$

& $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

Note: Since the market is complete, for any $\mathcal{F}_{n+1}$ meas. R.V. $\nearrow X_{n+1}$

$\exists$ an $\mathcal{F}_n$ meas R.V Δ_n $\ni$ $\boxed{X_{n+1} = \Delta_n \cdot S_{n+1}}$

So by completeness know $\exists$ Δ_n adapted $\ni$ $X_{n+1} = \Delta_n \cdot S_{n+1}$

Need: (1) $X_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

& (2) $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

(Note (2) $\Rightarrow$ (1) since $X_{n+1} = \Delta_n \cdot S_{n+1}$)

Prove (2): know $\tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n$ (linear)

Hence $D_n X_n = \tilde{E}_n\left(D_{n+1} X_{n+1}\right) - D_{n+1}\, \tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right)$

$D_n \Delta_{n-1} \cdot S_{n}$

$$= \sum_{i=0}^{d} \Delta_n^i \, \tilde{E}_n\left(D_{n+1} S_{n+1}^i\right)$$

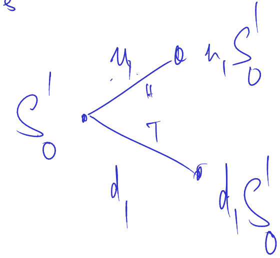
$$(\text{Def of RWM}) \quad = \sum_{i=0}^{d} \Delta_n^i \left(D_n S_n^i\right) = D_n \Delta_n \cdot S_n$$

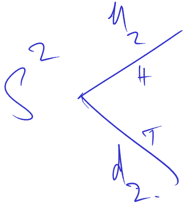
$$\Rightarrow \quad \Delta_{n-1} \cdot S_{n} = \Delta_n \cdot S_n \quad \Rightarrow \quad 2$$

QED

Question 7.17. *Consider a market consisting of a bank with interest rate r, and two stocks with price processes S^1, S^2. At each time step we flip two independent coins. The price of the i-th stock ($i \in \{1,2\}$) changes by factor u_i, or d_i depending on whether the i-th coin is heads or tails. When is this market arbitrage free? When is this market complete?*

2 stocks





(2 indep coins flipped at each time)

Guess: Arb free & complete $\iff$ $\left.\begin{array}{l} 0 < d_1 < 1+r < u_1 \\ \& \ 0 < d_2 < 1+r < u_2. \end{array}\right\}$

doesn't work

Claim ______ will $\Rightarrow$ arb free (not completeness)

Claim: this Market can $\boxed{\text{NEVER}}$ be made complete & arb free

Consider what happens after the first time period:

Look for the RNM. 2 coins $\longrightarrow$ outcomes HH, HT, TH, TT.

RNM probabilities: $\tilde{p}(HH)$, $\tilde{p}(HT)$, $\tilde{p}(TH)$, $\tilde{p}(TT)$.

Want $\frac{1}{(1+r)} \tilde{E} S_1^1 = S_0^1 \iff \tilde{E} S_1^1 = (1+r) S_0^1$

& $\frac{1}{1+r} \tilde{E} S_1^2 = S_0^2 \iff \tilde{E} S_1^2 = (1+r) S_0^2$

$$\Leftrightarrow (1+r)\cancel{S_0} = u_1 \cancel{S_0}\left(\tilde{p}(HH)+\tilde{p}(HT)\right) + d_1 \cancel{S_0}\left(\tilde{p}(TH)+\tilde{p}(TT)\right) \quad (1)$$

&

$$(1+r)\cancel{S_0^2} = u_2 \cancel{S_0^2}\left(\tilde{p}(HH)+\tilde{p}(TH)\right) + d_2 \cancel{S_0^2}\left(\tilde{p}(HT)+\tilde{p}(TT)\right) \quad (2)$$

$$\tilde{p}(HH)+\tilde{p}(HT)+\tilde{p}(TH)+\tilde{p}(TT) = 1 \quad (3)$$

3 equations.

Unknowns: $\tilde{p}(HH), \tilde{p}(HT), \tilde{p}(TH), \tilde{p}(TT)$ (4 unknowns!)

Solve with the constraints

$$0 \cancel{\leq} \tilde{p}(HH) \cancel{\leq} 1$$

$$0 \cancel{\leq} \tilde{p}(HT) \cancel{\leq} 1$$

$$0 \cancel{\leq} \tilde{p}(TH) < 1 \quad \& \quad 0 < \tilde{p}(TT) < 1$$

Ill constrained system $\rightarrow$ if solutions exist they will NEVR be unique!
(Market can never be complete & arb free)

Q: Say $0 < d_1 < 1+r < u_1$
& $0 < d_2 < 1+r < u_2$ } Does this $\Rightarrow$ existence of a RNM.

Yes: Let $\tilde{p}_1 = \dfrac{1+r-d_1}{u_1-d_1}$, $\tilde{q}_1 = \dfrac{u_1-(1+r)}{u_1-d_1}$

$\tilde{p}_2 = \dfrac{1+r-d_2}{u_2-d_2}$, $\tilde{q}_2 = \dfrac{u_2-(1+r)}{u_2-d_2}$

& Choose $\tilde{p}(HH) = \tilde{p}_1 \cdot \tilde{p}_2$

$\tilde{p}(HT) = \tilde{p}_1 \; \tilde{q}_2$

$\tilde{p}(TH) = \tilde{q}_1 \; \tilde{p}_2$

$\tilde{p}(TT) = \tilde{q}_1 \; \tilde{q}_2$ & check this gives a RNM.

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(1) $V \cap \bar{Q} = \{0\} \iff \exists\, \hat{n} \in \mathring{Q}$ s.t. $\hat{n} \perp V$.

(Stated but did not prove) (Hyp sep thm)

(2) Say $\exists$ a RWM NTS no arb

Say $X_0 = 0$, $X_N \geq 0$. Know $(D_n X_n)$ is a mg under $\tilde{P}$

$\Rightarrow E D_N X_N = E D_0 X_0 = 0$

$\Rightarrow X_N D_N \geq 0$ & $E(D_N X_N) = 0 \Rightarrow D_N X_N = 0$