Last time: 1^{st} fund thm: No arb $\iff \exists$ a RNM

Pf from last time: Assume no arb. NTS $\exists$ a RNM.

Going to constr $\tilde{P}$ with PMF $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\, \tilde{p}_2(\omega_1, \omega_2) \cdots \tilde{p}_N(\omega_1 \cdots \omega_N)$.

Find $\tilde{p}_n$. $\longrightarrow$ Fix $\omega' = (\omega_1, \cdots \omega_n)$

$$\text{Let } V = \left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \;\middle|\; \underbrace{\Delta_n(\omega') \cdot S_n(\omega') = 0}_{\text{Net worth } 0 \text{ at time } n} \right\}$$

j^{th} coordinate = net worth at time $n+1$ if $(n+1)^{th}$ die roll is j.

You check: $V \subseteq \mathbb{R}^M$ is a subspace (HW, please check)

$$\overline{Q} = \{v \in \mathbb{R}^M \mid v_i \geq 0\}, \quad \mathring{Q} = \{v \in \mathbb{R}^M \mid v_i > 0\}$$

Note: No arb $\Rightarrow V \cap \overline{Q} = \{0\}$.

Separation lemma $\Rightarrow \exists\, \hat{n} \in \mathring{Q}$ + $|\hat{n}| = 1$ & $\hat{n} \perp V$

(i.e. $\hat{n} \cdot v = 0 \; \forall v \in V$).

Use $\hat{n}$ to define $\tilde{p}_{n+1}(\omega', j)$:

$$\boxed{\tilde{p}_{n+1}(\omega', j) = \frac{\hat{n}_j}{\sum_{i=1}^{M} \hat{n}_i}}$$

$$\Rightarrow \tilde{E}_n(\Delta_n \cdot S_{n+1})(\omega') = \sum_{j=1}^{M} \Delta_n(\omega') \cdot S_{n+1}(\omega', j) \cdot \tilde{p}_{n+1}(\omega', j)$$

$$= \frac{1}{\sum_{i=1}^{M} \hat{n}_i} \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \cdot \hat{n}$$

$\perp V$.

$\in V$

$= 0$

Lemma from last time $\Rightarrow$ $\tilde{P}$ is a RNM.

QED

7.3. Second fundamental theorem.

Definition 7.11. A market is said to be *complete* if every derivative security can be hedged.

Theorem 7.12. *The market defined in Section 7.1 is complete and arbitrage free if and only if there exists a* unique *risk neutral measure.*

(i.e. For every security $\exists$ a replicating portfolio)

Lemma 7.13. *The market is complete if and only if for every $\mathcal{F}_{n+1}$-measurable random variable X_{n+1}, there exists a (not necessarily unique) $\mathcal{F}_n$ measurable random vector $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ such that $X_{n+1} = \Delta_n \cdot S_{n+1}$.*

Pf: Say first $\forall X_{n+1}$ $\exists \Delta_n$ s.t. $X_{n+1} = \Delta_n \cdot S_{n+1}$

(take position Δ_n at time n
Wealth at time $n+1$ $= \Delta_n \cdot S_{n+1}$)

Consider any security that pays G_N at time N.

NTS $\exists$ a rep portfolio. $\longrightarrow$ $\exists$ a self fin trading strat with final wealth G_N.

1. By assumption $\exists$ Δ_{N-1} ($\mathcal{F}_{N-1}$ meas) + $\Delta_{N-1} \cdot S_N = G_N$.
2. $\Delta_{N-1} \cdot S_{N-1}$ is $\mathcal{F}_{N-1}$ meas $\Rightarrow$ $\exists$ Δ_{N-2} ($\mathcal{F}_{N-2}$-meas) + $\Delta_{N-2} \cdot S_{N-1} = \Delta_{N-1} \cdot S_{N-1}$

$\vdots$

(n) Backward induction: $\forall n, \exists \Delta_n$ ($\mathcal{F}_n$-meas) $\ni$ $\overbrace{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}^{\text{self financing condition.}}$

& $\Delta_{N-1} \cdot S_N = G_N$.

$\Rightarrow (\Delta_n)$ gives me a self-financing trading strat that replicates the security. ✓

Pf of converse is similar (please check!)

Proof of Theorem 7.12 NTS complete + arb free $\iff$ unique RNM.

Recall how we constructed $\tilde{P}$ in the first fund thm.

Fix n, $\omega' = (\omega_1, \dots, \omega_n)$.

$$V = \left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \;\middle|\; \Delta_n(\omega') \cdot S_n(\omega') = 0 \right\}$$

How did we construct RNM: Picked $\hat{n} \perp V$ & $\hat{n} \in \mathring{Q}$ (all +ve coordites)

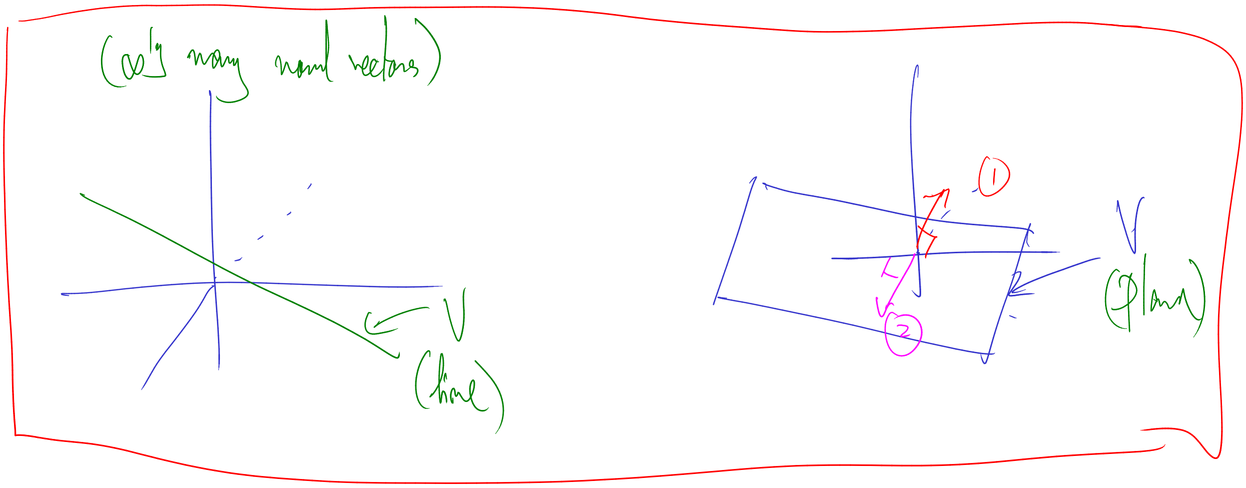
Note any $\hat{n}$ s.t. $\hat{n} \perp V$ & $\hat{n} \in \mathring{Q}$ gives a RNM

by $\tilde{f}_n(\omega', j) = \dfrac{\hat{n}_j}{\sum_{i=1}^{M} \hat{n}_i}$

Hence Unique RNM $\iff$ $\exists$ a unique $\hat{n} \in \mathring{Q}$ s.t. $|\hat{n}| = 1$ & $\hat{n} \perp V$

$$\iff \dim(V) = M - 1 \quad \& \quad \underbrace{V \cap \bar{Q} = \{0\}}_{\text{no arbitrage.}}$$

$$\iff \mathbb{R}^M = \operatorname{span}\left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \;\middle|\; \cancel{\Delta_n(\omega') \cdot S_n(\omega') = 0} \right\}$$



$\Longleftrightarrow$ any δ_{n+1} RV can be written in the form $\Delta_n \circ S_{n+1}$ & $\boxed{\text{no amb}}$

$\Updownarrow$

completeness (by lemma).

$\Longleftrightarrow$ completeness & no arb

Q.E.D.