Last time : d stocks. Price : $S^1 \cdots S^d$ (adapted processes)

M. M. S^0 (predictable)

$$S^0_{n+1} = (1 + r_n) S^0_n \qquad (r_n \longrightarrow \text{interest rate, adapted})$$

Discount factor : $D_n = \dfrac{1}{S^0_n}$ $\quad (\Leftrightarrow D_n S^0_n = 1)$

Self financing : $\Delta_n = (\Delta^0_n, \cdots \Delta^d_n)$ (position in the $d+1$ assets)

$$\Delta_n \cdot S_n = \text{wealth at time } n = \sum_0^d \Delta^i_n S^i_n .$$

Self fin : $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

7.2. **First fundamental theorem of asset pricing.**

Definition 7.2. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Definition 7.3. We say $\tilde{\boldsymbol{P}}$ is a *risk neutral measure* if $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ and $\tilde{\boldsymbol{E}}_n(D_{n+1}S^i_{n+1}) = D_n S^i_n$ for every $i \in \{0, \dots, d\}$.

Theorem 7.4. *The market defined in Section 7.1 is arbitrage free if and only if there exists a risk neutral measure.*

Lemma 7.5. *If $\tilde{\boldsymbol{P}}$ is a risk neutral measure, then the discounted wealth of any self financing portfolio is a $\tilde{\boldsymbol{P}}$-martingale.*

Proof that existence of a risk neutral measure implies no-arbitrage. (last time).

did last time.

Goal now: No arb $\Longrightarrow$ $\exists$ a RNM.

Lemma 7.6. *Suppose the market has no arbitrage, and X is the wealth process of a self-financing portfolio. If for any n, $X_n = 0$ and $X_{n+1} \geqslant 0$, then we must have $X_{n+1} = 0$ almost surely.*

Pf: Say we had $X_n = 0$, $X_{n+1} \geqslant 0$ & $P(X_{n+1} > 0) > 0$.

then move all \$ to bank

Get $\geqslant 0$ wealth at time N

& $P(0 < \text{wealth at time } N) > 0$

Not allowed by the no arb assumption

QED

Lemma 7.7. *Suppose we find an equivalent measure $\tilde{\boldsymbol{P}}$ such that whenever $\Delta_n \cdot S_n = 0$, we have $\tilde{\boldsymbol{E}}_n(\Delta_n \cdot S_{n+1}) = 0$, then $\tilde{\boldsymbol{P}}$ is a risk neutral measure.*

Pf: NTS $\tilde{E}_n(D_{n+1} S^i_{n+1}) = D_n S^i_n \quad \forall i \in \{\cancel{0}, 1, \dots d\}$

First show this for $i=1$ (For other i the proof is identical)

At time n → buy 1 share of 1^{st} stock. (Costs S^1_n)

→ sell S^1_n cash. $\left(= \frac{S^1_n}{S^0_n}\right.$ shares of M.M at time $n\left.\right)$.

i.e. Chose $\Delta_n = \left(-\frac{S^1_n}{S^0_n}, 1, 0 \dots 0 \right)$

Note $\Delta_n \cdot S_n = \left(-\frac{S_n^1}{S_n^0}, 1, 0 \cdots\right) \cdot \left(S_n^0, S_n^1, \cdots\right)$

$$= -\frac{S_n^1}{\cancel{S_n^0}} \cdot \cancel{S_n^0} + S_n^1 = 0$$

$\Rightarrow \Delta_n \cdot S_n = 0$. Hence, by assumption, $\tilde{E}_n(\Delta_n \cdot S_{n+1}) = 0$

$$\Rightarrow \tilde{E}_n\left(-\frac{S_n^1}{S_n^0} \cdot S_{n+1}^0 + 1 \cdot S_{n+1}^1 + 0 \cdots\right) = 0$$

$$\Rightarrow -\frac{S_n^1}{S_n^0} \cdot S_{n+1}^0 + \tilde{E}_n S_{n+1}^1 = 0$$

$$\Rightarrow \tilde{E}_n S^1_{n+1} = \frac{S^1_n}{S^0_n} \cdot S^0_{n+1}$$

$$\boxed{\Rightarrow \tilde{E}_n \left(D_{n+1} S^1_{n+1} \right) = D_n S^1_n} \Rightarrow D_n S^1_n \text{ is a } \tilde{P} \text{ mg}$$

Repeat for all i $\Rightarrow D_n S^i_n$ is a $\tilde{P}$ mg $\forall i$

$\Rightarrow \tilde{P}$ is a RNM

QED.

Lemma 7.8. *Suppose $\tilde{p}$ is a probability mass function such that $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\tilde{p}_2(\omega_1, \omega_2) \cdots \tilde{p}_N(\omega_1, \dots, \omega_N)$. If X_{n+1} is $\mathcal{F}_{n+1}$-measurable, then*

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \sum_{i=1}^{M} \tilde{p}_{n+1}(\omega', j) X_{n+1}(\omega', j)\,, \qquad \text{where} \qquad \omega' = (\omega_1, \dots, \omega_n)\,, \omega = (\omega', \cancel{\omega_{n+1}}, \omega_{n+1}, \dots, \omega_N)$$

Lemma 7.9. *Define* $\bar{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i \geqslant 0 \ \forall i \in \{1, \ldots, M\}\}$*, and* $\mathring{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i > 0 \ \forall i \in \{1, \ldots, M\}\}$*. Let* $V \subseteq R^M$ *be a subspace.*

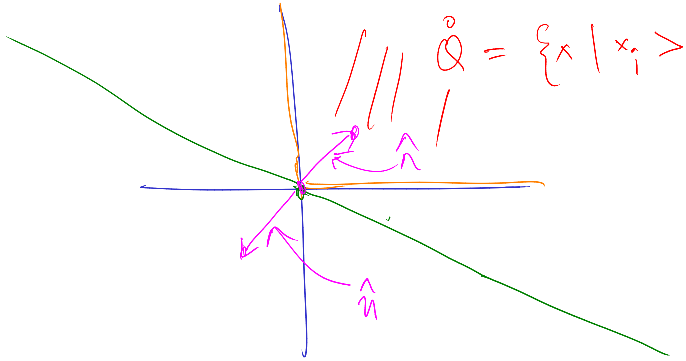
(1) $V \cap \bar{Q} = \{0\}$ *if and only if there exists* $\hat{n} \in \mathring{Q}$ *such that* $|\hat{n}| = 1$ *and* $\hat{n} \perp V$. ← $(\hat{n} \cdot v = 0 \ \forall v \in V)$

(2) *The normal vector* $\hat{n}$ *is unique if and only if* $\dim(V) = M - 1$.

Remark 7.10. This is a special case of the *Hyperplane separation theorem* used in convex analysis.

$\bar{Q} = \{x \mid x_i \geq 0\}$

$\mathring{Q} = \{x \mid x_i > 0 \ \forall i\}$



Proof of Theorem 7.4. Assume no arb. NTS $\exists$ a RNM.

① Construct the RNM using a PMF $\tilde{p}$ of the form

$$\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\,\tilde{p}_2(\omega_1,\omega_2) \cdots \tilde{p}_N(\omega_1, \cdots \omega_N)$$

& will find each $\tilde{p}_i$.

② Pick $n \in \{0, \cdots N-1\}$. Will find $\tilde{p}_n$

③ Know No arb $\Rightarrow$ No arb at time n.

i.e. If $\Delta_n \cdot S_n = 0$ & $\Delta_n \cdot S_{n+1} \geq 0$

& $\Delta_{n+1} \cdot S_{n+1} = \Delta_n \cdot S_{n+1}$

$\Rightarrow \underline{\Delta_{n+1} \cdot S_{n+1} = 0}$

Fix $\omega' = (\omega_1, \cdots \omega_n)$. Write $\Delta_{n+1} = \Delta_{n+1}(\omega', \omega_{n+1})$.

Let $V = \left\{ \left\| \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', 2) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \right\| \quad \Delta_n(\omega') \cdot S_n(\omega') = 0 \right\}$

Think of $V \subseteq \mathbb{R}^M$. Note V is a subspace of $\mathbb{R}^M$.

No arb: $V \cap \bar{Q} = \{0\}$.

$\bar{Q} = \{v \mid v_i \geq 0\}$

By lemma $\exists$ a $\boxed{\text{normal vector } \hat{n}}$ $\in \mathring{Q}$

Will use $\hat{n}$ to construct RNM.

$$\hat{n} = \hat{n}(\omega') = \begin{pmatrix} \hat{n}_1(\omega') \\ \vdots \\ \hat{n}_M(\omega') \end{pmatrix}.$$

Set: $\tilde{\phi}_n(\omega', j) = \dfrac{\hat{n}_j(\omega')}{\sum_i \hat{n}_i(\omega')}$

Claim (You check): $\underbrace{\tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right)} = 0$

(By lemma $\Rightarrow$ $\tilde{P}$ is a RWM $\Rightarrow$ QED.)