7. Fundamental theorems of Asset Pricing

7.1. Markets with multiple risky assets.

(1) $\Omega = \{1, \dots, M\}^N$ is a probability space representing N rolls of M-sided dies, and p is a probability mass function on Ω.
(2) The die rolls need not be i.i.d.
(3) Consider a financial market with $d+1$ assets $S^0, S^1, \dots, S^d$. (S_n^k denotes the price of the k-th asset at time n.)
(4) For $i \in \{1, \dots, d\}$, S^i is an adapted process (i.e. S_n^i is $\mathcal{F}_n$-measurable).
(5) The 0-th asset S^0 is assumed to be a *risk free* bank/money market:
   (a) Let r_n be an adapted process specifying the interest rate at time n.
   (b) Let $S_0^0 = 1$, and $S_{n+1}^0 = (1 + r_n) S_n^0$. (Note S^0 is predictable.)
   (c) Let $D_n = (S_n^0)^{-1}$ be the discount factor (D_n dollars at time 0 becomes 1 dollar at time n).
(6) Let $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ be the position at time n of an investor in each of the assets $(S_n^0, \dots, S_n^d)$.
(7) The wealth of an investor holding these assets is given by $X_n = \boxed{\Delta_n \cdot S_n} \stackrel{\text{def}}{=} \sum_{i=1}^{d} \Delta_n^i S_n^i$.

$i = 0$

Q: Say an investor has position $\underline{\Delta}_n$ on the d stocks & Money Market.

What does it mean for the investors wealth to be self financing?

(i.e. $\longrightarrow$ no cash flow injection / removal)

Binomial model

$$X_{n+1} = \Gamma_n S_{n+1} + (X_n - \Gamma_n S_n)(1+r)$$

Wealth at time n: $\sum_{i=0}^{d} \Delta_n^i S_n^i$

Say position at time n is $\underline{\Delta}_n$.

At time $(n+1) \longrightarrow$ my portfolio is now worth $\sum_{i=0}^{d} \Delta_n^i S_{n+1}^i = \Delta_n \cdot S_{n+1}$

Rebalance : (buy / sell some assets)

New position on the d assets is Δ_{n+1}

Wealth at time $n+1$ is $\Delta_{n+1} \cdot S_{n+1}$

Relevant

No cash injection / withdrawal $\Rightarrow$ $\boxed{\Delta_{n+1} \cdot S_{n+1} = \Delta_n \cdot S_{n+1}}$

Self financing condition.

Definition 7.1. Consider a portfolio whose positions in the assets at time n is Δ_n. We say this portfolio is *self-financing* if Δ_n is adapted, and $\underbrace{\Delta_n \cdot S_{n+1}}$ $= \Delta_{n+1} \cdot S_{n+1}$.

Note : Δ_n is $\mathcal{F}_n$ meas.

Δ_{n+1} is $\mathcal{F}_{n+1}$ meas.

7.2. **First fundamental theorem of asset pricing.**

Definition 7.2. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $\boxed{X_0 = 0}$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Definition 7.3. We say $\tilde{\boldsymbol{P}}$ is a *risk neutral measure* if $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ and $\tilde{\boldsymbol{E}}_n(D_{n+1}S^i_{n+1}) = D_n S^i_n$ for every $i \in \{0, \dots, d\}$.

Theorem 7.4. *The market is arbitrage free if and only if there exists a risk neutral measure.*

Proof that existence of a risk neutral measure implies no-arbitrage.

Recall: $S^0_{n+1} = (1 + r_n) S^0_n$ (bank)

$$\boxed{D_n \stackrel{\text{def}}{=} \frac{1}{S^0_n}}$$

Note $\tilde{E}_n(D_{n+1} S^0_{n+1}) = 1 = D_n S^0_n$

Recall: $\tilde{P}$ is equivalent to P if $P(A) = 0 \iff \tilde{P}(A) = 0$

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Pf of FTAP 1 $\Leftarrow$: Assume $\exists$ a RNM.

NTS $\longrightarrow$ No arb.

Pf: X = wealth process of a self fin portfolio.

$X_n = \Delta_n \cdot S_n$ & $\boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$

$\boxed{\text{Q: Is } D_n X_n \text{ a } \tilde{P} \text{ mg?}}$

Yes: Check: $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n(D_{n+1} \, \Delta_{n+1} \cdot S_{n+1})$

(D_n is predictable)

$$= D_{n+1} \tilde{E}_n \left(\Delta_{n+1} \cdot S_{n+1} \right)$$

$$\overset{\text{self financing}}{=} D_{n+1} \tilde{E}_n \left(\Delta_n \cdot S_{n+1} \right)$$

$$= D_{n+1} \tilde{E}_n \left(\sum_{i=0}^{d} \Delta_n^i S_{n+1}^i \right)$$

$$= \sum_{i=0}^{d} \Delta_n^i \tilde{E}_n \left(\underbrace{D_{n+1} S_{n+1}^i} \right) \overset{\text{RNM}}{=} \sum_{i=0}^{d} \Delta_n^i D_n S_n^i$$

$$= D_n \left(\Delta_n \cdot S_n \right) = D_n X_n$$

QED (Claim)

Have shown: X_n = wealth process of a self fin Port

$\Rightarrow D_n X_n$ is a $\tilde{P}$ mg.

Pf there is No Arb:

Say $X_0 = 0$, X is wealth proc of a self fin port.

$X_N \geq 0$.

$\Rightarrow D_n X_n$ is a $\tilde{P}$ mg. $\Rightarrow \tilde{E}(D_N X_N) \overset{mg}{=} \tilde{E}(D_0 X_0) = 0$

$\Rightarrow D_N X_N = 0$ a.s. ($\because D_N X_N \geq 0$)

$$\Rightarrow \quad X_N = 0 \quad \text{a.s} \qquad (\because D_N > 0).$$

QED.