Last time : $G = (G_0, \dots, G_N)$.

Snell super mg envelope : $\underline{V_N = G_N}$, $V_n = \max\{G_n, E_n V_{n+1}\}$.

Last time ① V is the smallest super mg s.t. $V \geq G$.

Last time ② $\tau^* = \min\{n \mid V_n = G_n\}$. τ^* solves the $\boxed{\text{optimal stopping problem for } G.}$

i.e. $E G_{\tau^*} \geq E G_\tau$ for all finite stopping times τ.

(τ^* is a stopping time).

Theorem 6.81. *Let $V = M - A$ be the Doob decomposition for V, and define $\tau^* = \max\{n \mid A_n = 0\}$. Then τ^* is a stopping time and is the largest solution to the optimal stopping problem for G.*

$M \longrightarrow M_g$

$A \rightarrow$ pred inc & $A_0 = 0$

Note $\{\tau^* \le n\} = \{A_{n+1} > 0\} \in \mathcal{F}_n$

($\because A$ is inc) ($\because A$ is pred)

i.e.: NTS ① $E G_{\tau^*} \geq E G_\tau$ $\forall$ finite stopping times τ.

& ② If σ^* is a soln to the opt stopping problem for G then $\tau^* \leq \sigma^*$.

Pf: ① Claim $\boxed{V_{\tau^*} = M_{\tau^*} = G_{\tau^*}}$

ⓐ If $\tau^* = N \longrightarrow V_N = G_N = M_N$. (nothing to check).

(b) Say $n < N$.

$$\mathbb{1}_{\{\tau^* = n\}} E_n V_{n+1} = E_n \left(M_{n+1} - A_{n+1} \right) \mathbb{1}_{\{\tau^* = n\}}$$

$$= \mathbb{1}_{\{\tau^* = n\}} M_n - \mathbb{1}_{\{\tau^* = n\}} A_{n+1}$$

(When $n = \tau^*$
$A_n = 0 \Rightarrow M_n = V_n$
& $A_{n+1} > 0$)

$$< \mathbb{1}_{\{\tau^* = n\}} V_n$$

But $V_n = \max \{ E_n V_{n+1}, G_n \}$

We've shown when $n = \tau^*$, $V_n > E_n V_{n+1}$

$\Rightarrow V_n = G_n$. $\quad$ ($A_n = 0 \Rightarrow M_n = V_n = G_n$

when $n = \tau^*$). QED.

Hence we know $V_{\tau^*} = M_{\tau^*} = G_{\tau^*}$.

② Check τ^* is optimal: $\forall \tau$, $E G_\tau \le E M_\tau$ $\quad$ ($\because M \ge V \ge G$)

$$\overset{OST}{=} E M_0 \overset{OST}{=} E M_{\tau^*} = \underline{E G_{\tau^*}}$$

$\Rightarrow \tau^*$ is optimal.

(3) NTS $\underline{\tau^*}$ is the largest sol to the optimal stopping problem.

Say $\sigma^* \geq \tau^*$ is a solution to the optimal stopping problem for G.

& Say $P(\sigma^* > \tau^*) > 0$.

Then $$EG_{\sigma^*} \leq EV_{\sigma^*} = E(M_{\sigma^*} - A_{\sigma^*})$$

$$= EM_{\sigma^*} - EA_{\sigma^*}$$

$$\overset{OST}{=} EM_{\tau^*} - \underline{EA_{\sigma^*}}$$

$$= E G_{\tau^*} - E A_{\sigma^*}$$

Note $\sigma^* \geq \tau^* \Rightarrow A_{\sigma^*} \geq A_{\tau^*} = 0$

Also, $P(\sigma^* > \tau^*) > 0$

$\Rightarrow P(A_{\sigma^*} > 0) > 0$

$\Rightarrow E A_{\sigma^*} > 0$

$$< E G_{\tau^*}.$$

(by)

$\Rightarrow \sigma^*$ can not be optimal!.

QED.