Last time: Doob decomp: $X = \underbrace{M}_{\text{mg}} + \underbrace{A}_{\text{pred}}$, $A_0 = 0$

Super mg: $X = \underbrace{M}_{\text{mg}} - \underbrace{A}_{\substack{\text{pred} \\ \text{inc} \\ A_0=0}}$

$\Rightarrow$ By OST $\quad E_n X_\tau \leq X_{\tau \wedge n} \qquad$ (X is a super mg)

(τ is bdd)

Theorem 6.76 (Snell). *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}\,. \qquad (n < N).$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

V is called the Snell envelope of G.

Pf: ① $V_n \geq G_n$ $\quad \left(\because V_n = \max\{E_n V_{n+1}, G_n\}\right)$

② V is a super mg $\quad \left(\because V_n = \max\{\quad\} \geq E_n V_{n+1}\right)$

③ NTS V is the smallest super mg $\geq G$.

i.e. If W is a super mg, $W \geq G \Rightarrow W \geq V$.

Pf: Since $W_N \geq G_N = V_N \Rightarrow W_N \geq V_N$

Backward induction: Say $W_{n+1} \geq V_{n+1}$

W is a supermg $\Rightarrow$ $W_n \geq E_n W_{n+1} \geq \underline{E_n V_{n+1}}$

Also know $W_n \geq \underline{G_n}$

$$\Rightarrow \quad W_n \geq \max\{G_n, E_n V_{n+1}\} = V_n.$$

QED.

Proposition 6.77. *If W is any martingale for which $W_n \geqslant G_n$, and for one stopping time τ^* we have $\boldsymbol{E}W_{\tau^*} = \boldsymbol{E}G_{\tau^*}$, then we must have $W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$, and $W_{\tau^* \wedge n}$ is a martingale.* (V = Snell super mg envelope of G).

Theorem 6.78. *Let $\sigma^* = \min\{n \mid V_n = G_n\}$. Then σ^* is the minimal solution to the optimal stopping problem for G. Namely, $\boldsymbol{E}G_{\sigma^*} = \max_\sigma \boldsymbol{E}G_\sigma$ where the maximum is taken over all finite stopping times σ. Moreover, if $\boldsymbol{E}G_{\tau^*} = \max_\sigma \boldsymbol{E}G_\sigma$ for any other finite stopping time τ^*, we must have $\tau^* \geqslant \sigma^*$.* (Pf is the same as we had for American options, please check).

Remark 6.79. By construction $V_{\sigma^* \wedge n}$ is a martingale.

Pf: Note $\boxed{W_n \geq V_n}$ $\forall n$ ($\because$ W is a mg $\Rightarrow$ W is a super mg & given $W \geq G \Rightarrow W \geq V$).

Given $E\, W_{\tau^*} = E\, G_{\tau^*}$ & $W_{\tau^*} \geq G_{\tau^*} \Rightarrow W_{\tau^*} = G_{\tau^*}$

Since $W \geq V \geq G \Rightarrow W_{\tau^*} = V_{\tau^*} = G_{\tau^*}$.

Claim : $W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$.

(Intuition : Know $W_{\tau^* \wedge n} \geq V_{\tau^* \wedge n}$.

$$\underbrace{\underbrace{W_{\tau^* \wedge n}}_{\text{Mg}} \quad \underbrace{V_{\tau^* \wedge n}}_{\text{super mg}}}_{\text{Guess}} \Rightarrow \text{Equality.}$$)

Pf: Backward Induction : ① $W_{\tau^* \wedge N} = V_{\tau^* \wedge N}$ $(\because \tau^* \leq N)$ ✓

② Assume for some n, $\underbrace{W_{\tau^*\wedge(n+1)}}_{mg} = \underbrace{V_{\tau^*\wedge(n+1)}}_{\text{super mg}}$. ($\&$ $W \geq V$),

$$\Rightarrow E_n\left(W_{\tau^*\wedge(n+1)}\right) = E_n V_{\tau^*\wedge(n+1)}$$

(OST)
$$\Rightarrow W_{\tau^*\wedge n} = E_n V_{\tau^*\wedge(n+1)} \leq V_{\tau^*\wedge n}.$$

Since $W_{\tau^*\wedge n} \geq V_{\tau^*\wedge n}$ $\Rightarrow$ $W_{\tau^*\wedge n} = V_{\tau^*\wedge n}$. QED

Theorem 6.80. *For any $k \in \{0, \dots, N\}$, let $\sigma_k^* = \min\{n \geqslant k \mid V_n = G_n\}$. Then $\boldsymbol{E}_k G_{\sigma_k^*} = \max_{\sigma_k} \boldsymbol{E}_k G_{\sigma_k}$, where the maximum is taken over all finite stopping times σ_k for which $\sigma_k \geqslant k$ almost surely.*

i.e.

$E_k G_{\tau_k^*} \overset{a.s}{\geq} E_k G_{\tau_k}$ for all finite stopping times τ_k such that $\tau_k \geq k$ a.s.

$\boxed{\text{On HW}}$

Theorem 6.81. *Let $V = M - A$ be the Doob decomposition for V, and define $\tau^* = \max\{n \mid A_n = 0\}$. Then τ^* is a stopping time and is the* largest *solution to the optimal stopping problem for G.*

Pf: ① τ^* is a stopping time because A is predictable & inc (Intuition).

Pf: $\{\tau^* \leq n\} = \{A_{n+1} > 0\} \in \mathcal{F}_n$ ($\because$ A is predictable).

② NTS τ^* solves the optimal stopping problem

i.e. NTS. $E G_{\tau^*} \geq E G_\tau$ $\forall$ finite stopping time τ.

Pf: $A_{\tau^*} = 0 \Rightarrow V_{\tau^*} = M_{\tau^*} \boxed{= G_{\tau^*}}$ (Why)

I found a better shorter proof of this without relying on what I did for American options. I'll present the proof next class

$\Rightarrow E G_\tau \le E V_\tau \le E M_\tau \qquad (M \ge V \ge G)$

$$\overset{OST}{=} M_0 \overset{OST}{=} E M_{\tau^*} = E G_{\tau^*}.$$

$\Rightarrow \tau^*$ is a sol to the optimal stopping problem

(3) NTS τ^* is the largest sol to the optimal stopping problem.

Say τ^* is a sol to the optimal stopping problem.

$\Rightarrow E G_{\tau^*} \ge E G_\tau \quad \forall$ finite stopping times τ.

$\Rightarrow$ (Know $G_{\tau^*} = V_{\tau^*}$ & $\boxed{A_{\tau^*} = 0}$ ← Why.

$A_{\tau^*} = 0 \implies \tau^* \leq \tau^*$

$\Rightarrow \tau^*$ is the largest sol to the optimal stopping problem.

QED

7. Fundamental theorems of Asset Pricing