American option $\longrightarrow$ $G = (G_0, \dots, G_N)$ (intrinsic value)

$V_n =$ AFP at time n.

$$\overbrace{D_n V_n \stackrel{\text{Doob}}{=} \underbrace{D_n X_n}_{\text{mg}} - \underbrace{A_n}_{\text{pred. inc.}}}$$

Guess: yes

Q: If $V_\sigma = G_\sigma$ must σ be an optimal exercise time?

Q: If $\underbrace{X_\sigma = G_\sigma}$ " " " " " " " ?

Guess 1: ~~No~~ Yes?

Know $X_n \geq V_n \geq G_n$

Guess 2: $\boxed{X_{\tau^*} = G_{\tau^*}} \implies V_{\tau^*} = G_{\tau^*}$

Guess $X_{\tau^*} = G_{\tau^*} \implies \tau^*$ is optimal.

NTS $\tilde{E}(D_{\tau^*} G_{\tau^*}) = \max_{\tau'} (\tilde{E} D_\tau G_\tau)$ $\forall$ finite stopping times τ

$\iff$ NTS $\tilde{E}(D_{\tau^*} V_{\tau^*}) = \tilde{E}(D_{\tau^*} G_{\tau^*}) \overset{NTS}{=} \max_\tau \tilde{E}(D_\tau G_\tau)$

$\tilde{E}(D_{\tau^*} V_{\tau^*}) = \tilde{E}(D_{\tau^*} X_{\tau^*})$

V_0 ~ APF of the american opt.

$$V_0 = V_0^{\tau^*} = \max_{\tau} V_0^{\tau} = \max_{\tau} \tilde{E}\left(D_\tau G_\tau\right)$$

$$V_0^{\tau^*} = \tilde{E}\left(D_{\tau^*} G_{\tau^*}\right)$$

V_0^{τ} = AFP at time 0 of an option with fixed maturity time τ & Payoff G_τ

$$= \frac{1}{\cancel{D_n}} \tilde{E}_{\cancel{n}} \left(\underline{D_\tau G_\tau}\right)$$

$\Leftrightarrow$ NTS $\tilde{E}(D_{\tau^*} V_{\tau^*}) = \tilde{E}(D_{\tau^*} G_{\tau^*}) \overset{NTS}{=} \max_{\tau} \tilde{E}(D_\tau G_\tau)$

$\| $

$\tilde{E}(D_{\tau^*} X_{\tau^*})$

(Assume: $V_{\tau^*} = X_{\tau^*} = G_{\tau^*}$)

Doob Decomp: $\tilde{D_n V_n} = D_n X_n - A_n.$

$n = \tau^*$: $D_{\tau^*} V_{\tau^*} = D_{\tau^*} X_{\tau^*} - A_{\tau^*}$

$\Rightarrow A_{\tau^*} = 0.$

Know OST $\tilde{E}(D_{\tau^*} X_{\tau^*}) = D_0 X_0 \overset{OST}{=} \tilde{E}(D_\tau X_\tau) \geq \tilde{E}(D_\tau G_\tau)$

$$\| $$

$$\tilde{E}(D_{\tau^*} G_{\tau^*})$$

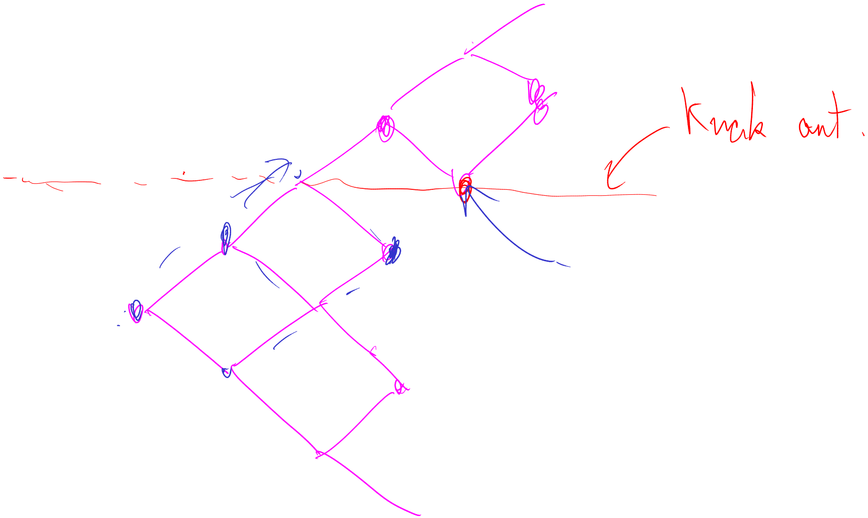
$$\Rightarrow \boxed{\tau^* \text{ is optimal}}$$

Guess 3: 2a $\rightarrow$ No

$V_0 = 1$ $\quad$ $V_1 = 1$ $\quad$ $V_2 = 0$ $\quad$ $V_3 = 0$

$r = 0$

$G_0 = 1$ $\quad$ $G_1 = 1$ $\quad$ $G_2 = 0$ $\quad$ $G_3 = 0$


Knock out.