Q3 Midterm:

Game $\begin{cases} \$1 & \text{prob } 1/2 \\ -\$1 & \text{prob } 1/2. \end{cases}$

Say now

Game $\begin{cases} +\$1 & \text{Prob } p \in (0, 1/2) \\ -\$1 & \text{Prob } 1-p. \end{cases}$

Prob you double your fortune when you stop?

Q3 Midterm:

Q: Play repeatedly.

Start with \$50.

Stop when lose all \$ or double initial stockpile.

Q1: Prob you double \$ when you stop $= \frac{1}{2}$ (OST)

Q2: Same game, start with \$1000 →

Prob (double) when you stop $= \frac{1}{2}$ (OST)

$$\alpha = \ln\left(\frac{1-p}{p}\right). \qquad P(\text{double}) = \frac{e^{\alpha S_0} - 1}{e^{\alpha(2S_0)} - 1}$$

(1) $S_0 \to$ small then $P(\text{double}) \approx 1/2$.

(2) $S_0 \to$ large then $P(\text{double}) \approx \dfrac{e^{\alpha S_0}}{e^{2\alpha S_0}} \approx \underline{e^{-\alpha S_0}}$

exponentially small!!

6.6. **Doob Decomposition and Optimal Stopping.**

Theorem 6.68 (Doob decomposition)**.** *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process that starts at* 0*. That is, if X is an adapted process there exists a unique pair of process M, A such that M is a martingale, A is predictable, $A_0 = 0$ and $X = M + A$.*

$$\left(E_n M_{n+1} = M_n \,, \qquad A_{n+1} \text{ is } \mathcal{F}_n \text{ meas.} \iff E_n A_{n+1} = A_{n+1} \right)$$

Pf: Guess from last time: If $X_n = M_n + A_n$.

$$E_n X_{n+1} = E_n M_{n+1} + E_n A_{n+1} = M_n + A_{n+1} =$$

$$= M_n + A_n - A_n + A_{n+1}$$

$$= X_n + (A_{n+1} - A_n).$$

$$\boxed{\implies \text{Should have } A_{n+1} = A_n + E_n X_{n+1} - X_n}$$

Know $A_0 = 0$, solve for A_n.

Pf: Let $\underline{A_0 = 0}$. $A_{n+1} = \underbrace{A_n + E_n X_{n+1} - X_n}$.

& Let $M_n = X_n - A_n$.

① A is predictable $\left(\because A_{n+1} = \underbrace{A_n}_{\mathcal{F}_n \text{ meas (ind)}} + \underbrace{E_n X_{n+1} - X_n}_{\mathcal{F}_n\text{-meas}} \right)$

② $A_0 = 0$. (Known)

③ NTS M is a mg.

$$E_n(M_{n+1}) = E_n(X_{n+1} - A_{n+1}) = E_n X_{n+1} - A_{n+1} \quad (\because A \text{ pred}).$$

$$= \cancel{E_n X_{n+1}} - (A_n + \cancel{E_n X_{n+1}} - X_n)$$

$$= X_n - A_n = M_n$$

QED. (Existence).

Pf of Uniqueness: Follow scratch. If $X_n = M_n + A_n$, we are forced to have

$A_0 = 0$ & $A_{n+1} = A_n + (E_n X_{n+1} - X_n)$. $\longleftarrow$ solving gives A_n uniquely ($X = M + A$ gives M uniquely) QED

Definition 6.69. We say an adapted process M is a *super-martingale* if $\boldsymbol{E}_n M_{n+1} \leqslant M_n$. (life is a super mg)

Definition 6.70. We say an adapted process M is a *sub-martingale* if $\boldsymbol{E}_n M_{n+1} \geqslant M_n$.

Example 6.71. The discounted arbitrage free price of an American option is a *super-martingale* under the risk neutral measure.

When pricing American options, we knew $D_n V_n$ is a $\tilde{P}$ super mg

& used Doob to write $D_n V_n = \underbrace{M_n}_{\text{mg}} - \underbrace{A_n}_{\text{predictable, } A_0 = 0}$

& increasing

Proposition 6.72. *If X is a super-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M - A$.*

Proposition 6.73. *If X is a sub-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M + A$.*

(Similar → you check)

Pf: $X \to$ super mg.

By Doob, $X = M - A$, $A_0 = 0$, M a mg & A predictable.

$$\Rightarrow \underbrace{E_n X_{n+1}} = E_n M_{n+1} - E_n A_{n+1} = M_n - A_{n+1} \quad (1)$$

Also know $E_n X_{n+1} \le X_n = M_n - A_n \quad (2)$

(super mg)

① & ② $\Rightarrow$ $\cancel{M_n} - A_{n+1} = E_n X_{n+1} \leq X_n = \cancel{M_n} - A_n$

$\Rightarrow A_n \leq A_{n+1} \rightarrow A_n$ is increasing QED.

Corollary 6.74. *If X is a super-martingale and τ is a bounded stopping time, then* $\boldsymbol{E}_n X_\tau \leqslant X_{\tau \wedge n}$.

Corollary 6.75. *If X is a sub-martingale and τ is a bounded stopping time, then* $\boldsymbol{E}_n X_\tau \geqslant X_{\tau \wedge n}$.

Know for a mg M & a bdd stopping time τ, $E_n M_\tau \overset{OST}{=} M_{\tau \wedge n}$

Q: Sub mg?

Q: Super mg?

Pf: Say X is a super mg, τ a bdd stopping time

WTS $E_n X_\tau \leq X_{\tau \wedge n}$

Use the Doob decomposition of X: $\exists M, A \ni X = M - A$, mn $M \to$ mg, $A \to$ pred & $\boxed{\text{inc}}$ & $A_0 = 0$

~~By OST~~: $E_n X_\tau = E_n(M_\tau - A_\tau)$

$\overset{OST}{=} M_{\tau\wedge n} - E_n A_\tau$

$\leq M_{\tau\wedge n} - \underbrace{E_n A_{\tau\wedge n}}$ ($\because$

$= M_{\tau\wedge n} - A_{\tau\wedge n}$ ($\because A$ is adapted & τ is a stopping time).

$= X_{\tau\wedge n}$. QED.

Note

$A_{\tau\wedge n} \leq A_\tau$

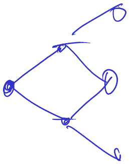
($\because A$ is inc)

Theorem 6.76 (Snell)**.** *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}\,.$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

AW Q3f:



10,000

n_0

$$X_{n+1} = X_n \pm 1$$

$X_{n_0} \in \mathbb{N}$.

$\text{Range}(X_n) \approx$ even or odd ~~Some~~ integers between $\underbrace{X_{n_0} \pm (n - n_0)}$

$$f_N(x) = \{ \mathbb{1}_{\{x \geq 0\}}$$

$(x \in (-N, N) + X_{n_0}$

$$f_n(x) = f_{n+1}\left(\underline{x+1}\right) \;\rule{2cm}{0.4pt}$$

$$+ \;\underline{\underline{f_{n+1}}}\left(\underline{x-1}\right) \;\rule{2cm}{0.4pt}$$