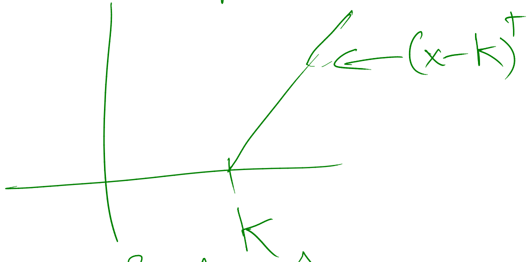
$r \geq 0$

Theorem 6.66. *Let g be a convex function with $g(0) = 0$ and let $G_n = g(S_n)$. Consider an American option with intrinsic value $G_n = g(S_n)$. You should never exercise this option early.*

i.e. N = maturity time is an optimal exercise time.

Corollary 6.67. *The arbitrage free price of an American call and European call are the same.*

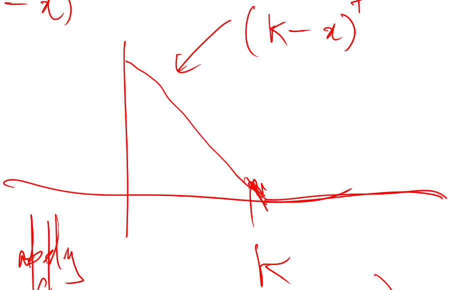
American call strike K: $g(x) = (x-K)^+$



(Convex & $g(0)=0$)
Thm applies

American put strike K:

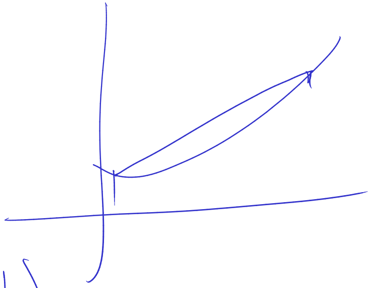
$g(x) = (K-x)^+$

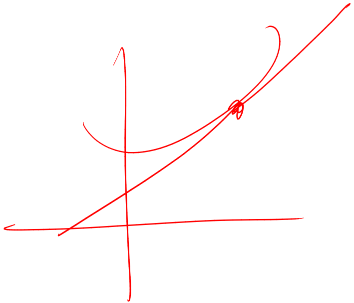


Thm does not apply
(convex, $g(0) \neq 0$).

Pf of thm: ① Convex: fn always lies below the chord (secant)

(⟺ fn always lies above the tangent)





② Recall Jensen's inequality

If g is convex & X is a RV then

$$Eg(X) \geq g(EX)$$ (on HW)

③ Conditional Jensen: $E_n g(X) \geq g(E_n X)$

Pf of Thm: $G_n = g(S_n)$ (intrinsic value)

$g(0) = 0$ & g convex.

AFP at time $N = g(S_N)$

AFP at time $n = V_n = \max\left\{ g(S_n), \frac{1}{D_n} \tilde{E}_n (D_{n+1} V_{n+1}) \right\}$

Let $\underline{\underline{n = N-1}}$: Claim: $V_n = \max\left\{ g(S_n), \frac{1}{D_n} \tilde{E}_n (D_{n+1} V_{n+1}) \right\}$

$= \max\left\{ g(S_n), \frac{1}{D_n} \tilde{E}_n \left(D_{n+1}\, g(S_{n+1}) \right) \right\}$

$$= \frac{1}{D_n} \tilde{E}_n\left(D_{n+1} g(S_{n+1})\right)).$$

(i.e. : $g(S_n) \leq \frac{1}{D_n} \tilde{E}_n\left(D_{n+1} g(S_{n+1})\right)$).

(Claim $\Rightarrow$ at time $n = N-1$, it is not in your interest to exercise. ~~cash out~~).

($\Rightarrow$ By backward induction $\Rightarrow$ at any time $\leq N$, it is not in your interest to exercise)

Pf of claim: NTS $g(S_n) \le \frac{1}{D_n}\tilde{E}_n\left(D_{n+1}\, g(S_{n+1})\right)$ ($\Rightarrow$ Claim $\Rightarrow$ thm $\Rightarrow$ QED)

Pf: $\frac{1}{D_n}\tilde{E}_n\left(D_{n+1}\, g(S_{n+1})\right) = \frac{D_{n+1}}{D_n}\tilde{E}_n\, g(S_{n+1})$ $\leftarrow$

Conditional Jensen

$$\ge \frac{D_{n+1}}{D_n}\, g\left(\tilde{E}_n S_{n+1}\right)$$

$$= \frac{D_{n+1}}{D_n}\, g\left(\frac{1}{D_{n+1}}\tilde{E}_n\left(D_{n+1} S_{n+1}\right)\right)$$

$$= \frac{D_{n+1}}{D_n}\, g\left(\frac{1}{D_{n+1}} D_n S_n\right)$$

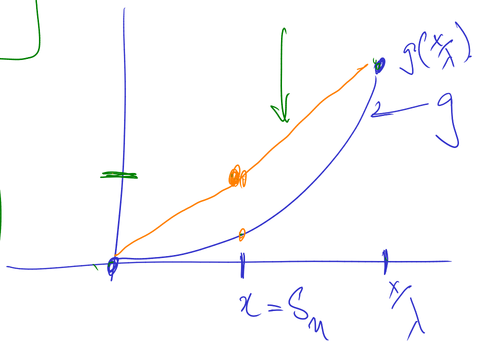
$(\because D_n S_n$ is a mg!)

(*) $$= \frac{D_{n+1}}{D_n}\, g\left(\frac{D_n}{D_{n+1}} S_n\right) \leftarrow$$

Let $\lambda = \frac{D_{n+1}}{D_n}$. $\boxed{\text{Interest rate} > 0 \implies \lambda \leq 1}$

slope $= \frac{g(x/\lambda)}{x/\lambda}$

(**) $\boxed{\text{By convexity } g(x) \leq x \frac{g(x/\lambda)}{x/\lambda} = \lambda g\left(\frac{x}{\lambda}\right)}$



$\therefore$ From $(*)$

$$\frac{1}{D_n}\tilde{E}_n D_{n+1}\, g(S_{n+1}) \geq \frac{D_{n+1}}{D_n}\, g\left(\frac{D_n}{D_{n+1}} S_n\right) \overset{(**)}{\geq} g(S_n) \,.$$

$\Rightarrow$ Claim $\Rightarrow$ QED!

6.6. **Doob Decomposition and Optimal Stopping.**

Theorem 6.68 (Doob decomposition)**.** *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process.* starting at 0

$X_n \to$ adapted process (X_n is $\mathcal{F}_n$ meas $\forall n$).

$M_n \to$ mg ($E_n M_{n+1} = M_n$)

$A_n \to$ predictable (A_{n+1} is $\mathcal{F}_n$ meas.).

Pf: X is adapted $\Rightarrow \exists!$ mg M & a predictable process A $\ni$

$X = M + A$. & $A_0 = 0$.

Pf scratch: Say $X = \underset{\substack{\uparrow \\ \text{mg}}}{M} + \underset{\substack{\uparrow \\ \text{prd.}}}{A}$. (Find M & A).

$$E_n X_{n+1} = E_n M_{n+1} + E_n A_{n+1}$$

$$= M_n + A_{n+1} = \underbrace{M_n + A_n}_{X_n.} - A_n + A_{n+1}$$

$$\Rightarrow E_n X_{n+1} = X_n + A_{n+1} - A_n$$

$$\Rightarrow A_{n+1} = A_n + (E_n X_{n+1} - X_n)$$

~~Def~~ Use this to define A_{n+1} (start with $A_0 = 0$)

Set $M_{n+1} = X_{n+1} - A_{n+1}$

Should give me the desired decomposition

(do backward $\Rightarrow$ QED Next time).