

Q1a] HW 9

$$V_{\text{call}} = \underline{G_N} \left[= (S_N - K)^+ \right]$$

$$D_n = (1+r)^{-n}$$

$$\frac{D_{n+1}}{D_n} = \frac{1}{1+r}$$

$$\rightarrow V_n = \max \left\{ (S_n - K)^+, \frac{1}{1+r} \left(\tilde{E}_n V_{n+1} \right) \right\}$$

$$V_n = f_n(S_n) \quad \cdot \quad (\text{Can be done for } n = N)$$

$$f_n(S_n) = V_n = \max \left\{ (S_n - K)^+, \frac{1}{1+r} \left(\tilde{p} f_{n+1}(u S_n) + \tilde{q} f_{n+1}(d S_n) \right) \right\}$$

$$V_n = f(S_n) \rightarrow \begin{cases} \underline{b}_N(s) = (s - K)^+ \\ \underline{b}_n(s) = \max \left\{ (s - K)^+, \frac{1}{1+r} \left(\tilde{p} \underline{b}_{n+1}(s_u) + \tilde{q} \underline{b}_{n+1}(s_d) \right) \right\} \end{cases}$$

$s \in \text{Range}(S_n)$

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Q: What is range S_n ?

$$C_n \rightarrow RV.$$

$$S_n : \Omega \rightarrow \mathbb{R}$$

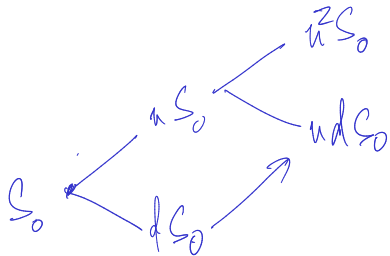
Q: What are all possible values $S_n(\omega)$ can take on $\forall \omega \in \Omega$.

$$\textcircled{1} \text{ Range}(S_0) = \{S_0\}$$

$$\textcircled{2} \text{ Range}(S_1) = \{uS_0, dS_0\}$$

$$\textcircled{3} \text{ Range}(S_{n+1}) = \{ \underline{u}s \mid s \in \text{Range}(S_n) \} \cup \{ \underline{d}s \mid s \in \text{Range}(S_n) \}$$

Can solve & get $\text{Range}(S_n) = \{ \underbrace{\binom{n}{k} u^k d^{n-k}}_{\text{coefficient}} \cdot S_0 \mid k \in \{0, \dots, n\} \}$



$d^2 S_0$

Q2: American ~~call~~ ^{option}: $g_N(x) = (x - K)^+$. $G_n = g_n(S_n)$
 $\hookrightarrow G_N = (S_N - K)^+$

Told: Every exercise policy is optimal.

Q: ~~find g_n~~ . Find finite diff eq^c for g_n .

Knows $J_N(s) = (s - K)^+$.

Say $n = N-1$. Say we know g_n 's.

$$V_n = \text{AFP at time } n. = \underline{f_n(S_n)}$$

Knows $V_n = f_n(S_n) = \max \left\{ \underbrace{g_n(S_n)}_{\text{Cash out}}, \underbrace{\frac{1}{1+r} \left(\tilde{P} f_{n+1}(uS_n) + \tilde{Q} f_{n+1}(dS_n) \right)}_{\text{wait}} \right\}$

If every exercise policy is optimal, expect

$$g_n(s_n) = \frac{1}{1+r} \left(\tilde{p} f_{n+1}(s_n^u) + \tilde{q} f_{n+1}(s_n^d) \right)$$

\swarrow
 \uparrow
 write in terms of g

$$\text{guess } f_{n+1} = g_{n+1}$$

← True when ~~at~~ ~~at~~ $n+1 = N$.

guess: $g_N(s) = (s - K)^+$

$$g_n(s) = \frac{1}{1+r} \left(g_{n+1}(u s) \tilde{p} + g_{n+1}(d s) \tilde{q} \right)$$

Prior the American option with intrinsic value $g = (G_0) - (G_N)$

$$G_k = g(S_k).$$

$$\rightarrow V_n = f_n(S_n) = \max \left\{ G_n, \frac{1}{1+r} \mathbb{E}_n \left[f_{n+1}(S_{n+1}) \right] \right\}$$

$$\text{(By choice of } G_n) = \frac{1}{1+r} \mathbb{E}_n \left[G_{n+1}(S_{n+1}) \right]$$

Q: Can you quickly show every exercise policy is optimal?

i.e. Show \forall ^{finite} stopping times

$$\mathbb{E} D_{\tau} G_{\tau} = G_0$$

Trick: X_n let Y be any RV.

let $X_n = E_n Y$ then X is a mg

$$\left(\text{Check: } E_n X_{n+1} = E_n (E_{n+1} Y) \stackrel{\text{Tower}}{=} E_n Y = X_n \right)$$

