Q1a] HW9

$$V_{AN} = G_N = (S_N - K)^+$$

$$D_n = (1+r)^{-n}$$

$$\frac{D_{n+1}}{D_n} = \frac{1}{1+r}$$

$$\rightarrow V_n = \max\left\{ (S_n - K)^+ , \frac{1}{1+r}\left(\tilde{E}_n V_{n+1}\right) \right\}$$

$V_n = f_n(S_n)$. (Can be done for $n = N$)

$$f_n(S_n) = V_n = \max\left\{ (S_n - K)^+ , \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(u S_n) + \tilde{q}\, f_{n+1}(d S_n) \right) \right\}$$

$$V_n = f(S_n) \rightarrow \begin{cases} f_N(s) = (s-K)^+ \\ f_n(s) = \max\left\{ (s-K)^+ , \frac{1}{1+r}\left(\tilde{p} f_{n+1}(su) + \tilde{q} f_{n+1}(sd)\right)\right\} \end{cases}$$

$$\boxed{s \in \text{Range}(S_N)}$$

$s \in \text{Range}(S_n)$

Q: What is range S_n?

$S_n \rightarrow$ RV.

$S_n : \Omega \longrightarrow \mathbb{R}$

Q: What are all possible values $S_n(\omega)$ can take on $\forall\, \omega \in \Omega$.

① $\text{Range}(S_0) = \{S_0\}$

② $\text{Range}(S_1) = \{uS_0, dS_0\}$

③ $\text{Range}(S_{n+1}) = \{\underline{\underline{u}}s \mid s \in \text{Range}(S_n)\} \cup \{\underline{d}s \mid s \in \text{Range}(S_n)\}$.

Can solve & get $\text{Range}(S_n) = \{\underline{u^k d^{n-k} S_0} \mid k \in \{0, \dots, n\}\}$.

$$\begin{tikzcd}
 & & u^2S_0 \\
 & uS_0 \arrow[ur, no head] \arrow[dr, no head] & \\
S_0 \arrow[ur, no head] \arrow[dr, no head] & & udS_0 \\
 & dS_0 \arrow[ur] &
\end{tikzcd}$$

$d^2 S_0$

Q2: American option ~~call~~: $g_N(x) = (x-K)^+$. $G_n = g_n(S_n)$

$G_N = (S_N - K)^+$

$\boxed{\text{Told}}$: $\boxed{\text{Every exercise policy is optimal.}}$

Q: ~~Find g_n.~~ Find finite diff eq^c for g_n

Knows $g_N(s) = (s-K)^+$.

Say $n = N-1$. Say we know g_n's.

V_n = AFP at time n = $f_n(S_n)$

Know $V_n = f_n(S_n) = \max\{g_n(S_n), \frac{1}{1+r}(\tilde{p} f_{n+1}(uS_n) + \tilde{q} f_{n+1}(dS_n))\}$

Cash out = wait

If every exercise policy is optimal, expect

$$g_n(S_n) = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(S_n u) + \tilde{q}\, f_{n+1}(S_n d)\right)$$

write in terms of g

$\boxed{\text{Guess } f_{n+1} = g_{n+1}}$ ← True when $n+1 = N$.

Guess: $g_N(s) = (s-K)^+$.

$$g_n(s) = \frac{1}{1+r}\left(g_{n+1}(us)\,\tilde{p} + g_{n+1}(ds)\,\tilde{q}\right)$$

Price the American option with intrinsic value $G = (G_0, \dots, G_N)$

$$G_k = g_k(S_k).$$

$$\to \quad V_n = f_n(S_n) = \max\left\{ G_n, \frac{1}{1+r} \tilde{E}_n \left(f_{n+1}(S_{n+1}) \right) \right\}.$$

(by choice of g_n) $= \frac{1}{1+r} \tilde{E}_n f_{n+1}(S_{n+1})$

Q: Can you quickly show every exercise policy is optimal?

i.e. Show $\forall$ finite stopping times $\boxed{\tilde{E} D_{\tau^*} G_{\tau^*} = G_0}$

Trick: X_n Let Y be any RV.

Let $X_n = E_n Y$ then X is a mg

(Check: $E_n X_{n+1} = E_n(E_{n+1} Y) \overset{\text{Tower}}{=} E_n Y = X_n$)