Last time: American option intrinsic value G_n.

Let $V_N = G_N$ & $V_n = \max\left\{ G_n , \frac{1}{D_n} \tilde{E}_n(D_{n+1} V_{n+1}) \right\}$

Then claim: ① The option can be replicated

② AFP at time n is V_n ←

& ③ Minimal optimal exercise time is $\sigma^* = \min\{ n \mid V_n = G_n \}$

Pf: Last time: ① $D_n V_n \geq \tilde{E}_n (D_{n+1} V_{n+1})$ ($\because D_n V_n$ is a $\tilde{P}$ super mg)

② Doob decomposition: $D_n V_n = \underbrace{D_n X_n}_{\tilde{P} \text{ mg}} - \underbrace{A_n}_{\text{predictable, inc, } \& \ A_0 = 0}$

(IOU)

③ Last time: Checked $A_{\tau^*} = 0$ $\left(\Rightarrow \mathbb{1}_{\{n \leq \tau^*\}} A_n = 0 \right)$.

④ Claim $X = \text{wealth}_n$ of the rep portfolio ($\Rightarrow$ Option can be replicated).
(proves)

(already did this. Review:

NTS: ⓐ $X_n \geq G_n$ ✓

& (b) For one finite stopping time $X_{\tau^*} = G_{\tau^*}$

Check (a): $D_n X_n = D_n V_n + A_n \Rightarrow X_n = \underbrace{V_n}_{V_n \geq G_n \text{ (by formula)}} + \underbrace{\frac{A_n}{D_n}}_{\geq 0}$

$\Rightarrow \quad X_n \geq V_n \geq G_n$.

(b) At time τ^*, claim $X_{\tau^*} = V_{\tau^*} = G_{\tau^*}$

(Pf: By def of τ^*, $V_{\tau^*} = G_{\tau^*}$. & know $A_{\tau^*} = 0$).

5. Check τ^* = minimal optimal exercise time.

τ^* is an optimal exercise time if $\tilde{E}(D_{\tau^*} G_{\tau^*}) = \max_{\tau} \tilde{E}(D_\tau G_\tau)$

(max over <u>all</u> finite stopping times τ)

Note $\tilde{E}(D_\tau G_\tau) \leq \tilde{E}(D_\tau X_\tau) \overset{OST}{=} D_0 X_0$ ($\because D_n X_n$ is a $\tilde{P}$ mg)

$\overset{OST}{=} \tilde{E}(D_{\tau^*} X_{\tau^*}) = \tilde{E}(D_{\tau^*} G_{\tau^*})$

$\Rightarrow$ <u>τ^* is optimal</u>!

Check τ^* is the minimal optimal exercise time.

Say $\tilde{\tau}^*$ is any optimal exercise time

$$\Rightarrow \tilde{E}\left(D_{\tilde{\tau}^*} G_{\tilde{\tau}^*}\right) = \max_{\tau} \tilde{E}\left(D_\tau G_\tau\right) = \tilde{E}\left(D_{\tau^*} G_{\tau^*}\right)$$

$(\because \tau^* \text{ is optimal})$.

$$= \tilde{E}\left(D_{\tau^*} X_{\tau^*}\right) \overset{OST}{=} D_0 X_0 \overset{OST}{=} \tilde{E}\left(D_{\tilde{\tau}^*} X_{\tilde{\tau}^*}\right)$$

$$\Rightarrow \tilde{E}\left(D_{\tilde{\tau}^*} G_{\tilde{\tau}^*}\right) = \tilde{E}\left(D_{\tilde{\tau}^*} X_{\tilde{\tau}^*}\right)$$

$$\Rightarrow \quad D_{\tau^*} G_{\tau^*} = \cancel{X} D_{\tau^*} X_{\tau^*} \qquad \left(\because \; G_{\tau^*} \leq X_{\tau^*} \text{ from above!} \right)$$

$$\Rightarrow \quad G_{\tau^*} = X_{\tau^*}$$

$$\Rightarrow \quad V_{\tau^*} = G_{\tau^*}$$

$$\Rightarrow \quad \nu^* \leq \tau^* \qquad \left(\because \; \nu^* = \text{first time } V_n = G_n \right)$$

$\Rightarrow$ ν^* is the <u>minimal</u> optimal exercise time.

Let's also check V_n = <u>AFP</u> at time n.

Case I: Say at time n we Buy one American option at price V_n.

Short cash/stock for $-V_n$

at time n $\begin{cases} \to 1 \text{ option } (+) \\ \to -V_n \text{ in cash/stock.} \end{cases}$

Say we chose to first trade this option at time τ. ($\tau \geq n$)

Say our cash/stock portfolio is worth Y_k at time k.

(Note $Y \to$ wealth of a self financing portfolio).

At time τ, wealth $= V_\tau - Y_\tau$

Note $\tilde{E}_n\left(D_\tau(V_\tau - Y_\tau)\right) = \tilde{E}_n\left(D_\tau V_\tau\right) - \tilde{E}_n\left(D_\tau Y_\tau\right)$

OST for super mg's (IOU) $\longrightarrow$ (OST) (OST since DY is a $\tilde{P}$ mg)

$$\leq D_n V_n - D_n Y_n$$

$$= D_n V_n - D_n V_n = 0$$

$$\Rightarrow \tilde{E}_n\left(D_\tau(V_\tau - Y_\tau)\right) \leq 0 \Rightarrow \boxed{\text{no arbitrage opportunity}}$$

Case II: Say at time n we sell ~~one~~ 1 options.

Wealth at time n $\begin{cases} V_n \text{ in stock / cash.} \\ \text{short 1 option (worth } V_n) \end{cases}$

Y_n = wealth of my portfolio in stock / cash.

Client cashes out the option when it is optimal for them

Say the client ~~ca~~ sells the option at time $\tau^* \vee n = \max\{\tau^*, n\}$.

Claim: No arb opportunity at time σ^*.

$$(*) \quad \ldots \text{but} \quad \left(D_{\sigma^* \vee n} V_{\sigma^* \vee n} - D_n V_n \right) = \left(D_{\sigma^* \vee n} X_{\sigma^* \vee n} - D_n X_n \right)$$

$$- \underbrace{\left(A_{\sigma^* \vee n} - A_n \right)}_{0 \;\; (\because A_n = 0 \;\forall\, n \le \sigma^*)}$$

$\Rightarrow$ at time $\sigma^* \vee n$ we have

$$E_n\left(D_{\sigma^* \vee n} Y_{\sigma^* \vee n} - D_{\sigma^* \vee n} V_{\sigma^* \vee n} \right) \underset{(*)\,\&\,\text{OST}}{=}$$

$$E_n\, D_{\tau^* \vee n}\, Y_{\tau^* \vee n} - D_n V_n$$

$$\overset{OST}{=} \cancel{q}\, D_n Y_n - D_n V_n = 0$$

$\Rightarrow$ No arbitrage opportunity.