Theorem 6.62. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N \,, \qquad V_n = \max\Big\{ \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_{n+1} V_{n+1}), G_n \Big\} \,, \qquad \sigma^* = \min\{ n \leqslant N \mid V_n = G_n \} \,.$$

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.

Remark 6.63. The above is true in any complete, arbitrage free market.

Remark 6.64. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{ \frac{1}{1+r} \Big(\tilde{p} V_{n+1}(\omega', 1) + \tilde{q} V_{n+1}(\omega', -1) \Big), G_n(\omega) \Big\} \,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'') \,, \quad \omega' = (\omega_1, \dots, \omega_n) \,.$$

Last time: Prove next this option can be replicated.

IOU: σ^* is the minimal optimal exercise time

IOU: V_n is the AFP.

Theorem 6.65. *Consider the ~~Binomial model~~ with* $0 < d < 1 + r < u$, *and a state process* $Y = (Y^1, \dots, Y^d)$ *such that* $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega'), \omega_{n+1})$, *where* $\omega' = (\omega_1, \dots, \omega_n)$, $\omega = (\omega', \omega_{n+1}, \dots, \omega_N)$, *and* $h_0, h_1, \dots, h_N$ *are* $N+1$ *deterministic functions. Let* $g_0, \dots, g_N$ *be* N *deterministic functions, let* $G_k = g_k(Y_k)$, *and consider an American option with intrinsic value* $G = (G_0, G_1, \dots, G_N)$. *The* pre-exercise price *of the option at time* n *is* $f_n(Y_n)$, *where*

$$f_N(y) = g_N(y) \quad \textit{for } y \in \operatorname{Range}(Y_N), \qquad f_n(y) = \max\Big\{ g_n(y), \frac{1}{1+r}\Big(\tilde p f_{n+1}(h_{n+1}(y, \overset{1}{\cancel{u}})) + \tilde q f_{n+1}(h_{n+1}(y, \overset{-1}{\cancel{d}}))\Big)\Big\}, \quad \textit{for } y \in \operatorname{Range}(Y_n).$$

The minimal optimal exercise time is $\sigma^* = \min\{n \mid f_n(Y_n) = g_n(Y_n)\}$.

Pf: Know AFP at time $n = V_n$, where

$$V_N = G_N, \qquad V_n = \max\Big\{ \frac{1}{D_n} \tilde{E}_n D_{n+1} V_{n+1},\ G_n \Big\}$$

$$= g_N(Y_N) = f_N(Y_N) \qquad = \max\Big\{ \frac{1}{1+r} \tilde{E}_n f_{n+1}(Y_{n+1}),\ G_n \Big\}$$

(Assume $V_{n+1} = f_{n+1}(Y_{n+1})$)

$$= \max\left\{ g_n(Y_n),\ \frac{1}{1+r}\tilde{E}_n f_{n+1}\left(h_{n+1}(Y_n, \omega_{n+1})\right) \right\}$$

indep lemma

$$= \max\left\{ g_n(Y_n),\ \frac{1}{1+r}\Big(f_{n+1}\big(h_{n+1}(Y_n, +1)\big)\tilde{p} + f_{n+1}\big(h_{n+1}(Y_n, -1)\big)\tilde{q} \Big) \right\}$$

Set $y = Y_n \Rightarrow$ done! QED.

m

Pay IOU's: $V_N = G_N$. $V_n = \max\{G_n, \frac{1}{D_n}\tilde{E}_n(D_{n+1} V_{n+1})\}$

$$\tau^* = \min\{n \mid V_n = G_n\}.$$

IOU: V_n = AFP (maybe next time)

τ^* = minimal optimal exercise time. } ← Prove (Hopefully).

Last time: American option can be replicated. } ← Review (+ fix typo)

Main Idea: Doob decomposition: Super mg = Mg − (predictable & inc).

Super mg : $E_n Y_{n+1} \le Y_n$

Note : $V_n = \max\left(G_n, \frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1})\right) \ge \frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1})$

$\Rightarrow D_nV_n \ge \tilde{E}_n(D_{n+1}V_{n+1}) \Rightarrow D_nV_n$ is a $\tilde{P}$ super mg.

2 Doob $\Rightarrow D_nV_n = M_n - A_n$: $M_n \to \tilde{P}$ ~~super~~ mg

$A_n \to$ Predictable inc. & $A_0 = 0$

Write $M_n = D_nX_n$.

$$\boxed{\Rightarrow D_nV_n = D_nX_n - A_n}$$

③ Last time X_n = wealth of a self fin portfolio

$(X_n \geq G_n \ \ldots)$

④ Claim $A_{\tau^*} = 0$ (A increasing & $A_0 = 0$

$\Rightarrow A_n = 0 \quad \forall n \leq \tau^*$)

Pf: $\mathbb{1}_{\{n < \tau^*\}} V_n = \frac{1}{D_n} \tilde{E}_n (V_{n+1} D_{n+1}) \mathbb{1}_{\{n < \tau^*\}}$

(by def of V_n & τ^*).

$$\Rightarrow \quad 1_{\{n < \tau^*\}} (D_n V_n) = 1_{\{n<\tau^*\}} \tilde{E}_n (D_{n+1} V_{n+1})$$

Also note $\quad \tilde{E}_n (D_{n+1} X_{n+1}) = D_n X_n \quad (\because D_n X_n \text{ is a } \tilde{P} \text{ mg})$

and $\quad \tilde{E}_n (A_{n+1}) = A_{n+1} \quad (\because A \text{ is predictable})$

Know $D_{n+1} V_{n+1} = D_{n+1} X_{n+1} - A_{n+1}$. mult by $1_{\{n<\tau^*\}}$ & take $\tilde{E}_n$

$$\Rightarrow \quad 1_{\{n<\tau^*\}} D_n V_n = 1_{\{n<\tau^*\}} \left[(D_n X_n) - A_{n+1} \right]$$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} D_n V_n = \mathbb{1}_{\{n<\tau^*\}} \Big[\underbrace{D_n X_n - A_n}_{D_n V_n} + A_n - A_{n+1} \Big]$$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} (A_n - A_{n+1}) = 0$$

$$\left.\begin{array}{l} \Rightarrow \mathbb{1}_{\{n<\tau^*\}} A_{n+1} = \mathbb{1}_{\{n<\tau^*\}} A_n \\ \text{Also know } A_0 = 0 \end{array}\right\} \Rightarrow A_{\tau^*} = 0$$

$$\Rightarrow \mathbb{1}_{\{n \leq \tau^*\}} A_n = 0$$

$$\boxed{(\text{i.e. } A_n = 0 \ \forall n \leq \tau^*)}$$

Claim: τ^* is the minimal optimal exercise time.

Recall: optimal exercise time τ: $\tilde{E}(G_\tau D_\tau) = V_0 = \max_\tau \tilde{E}(G_\tau D_\tau)$

IOU ① ~~Clearly~~ τ^* is an optimal exercise time.

IOU ② τ^* is the minimal optimal exercise time

(AFP)