Last time: American option $\longrightarrow$ intrinsic value $G. \longrightarrow (G_0, G_1, \cdots G_N)$

$\hookrightarrow$ Can exercise at any time (finite stopping time)

Payoff at time τ is G_τ

Q: Price $\longrightarrow \max\{V_0^\tau \mid \tau$ any finite stopping time$\}$

& V_0^τ = AFP at time 0 of the security with payoff G_τ at time $\underline{\tau}$.

Replication: ① Need a self fin portfolio with wealth X ∋

ⓐ $X_n \geq G_n$

ⓑ $X_{\tau^*} = G_{\tau^*}$ for one stopping time τ^*.

Theorem 6.62. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

Then V_n is the arbitrage free price, at time n *and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.*

Remark 6.63. The above is true in any complete, arbitrage free market.

Remark 6.64. In the Binomial model the above simplifies to:

$$V_{n\cancel{+1}}(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \ldots, \omega_n)\,.$$

Proof: V_n = AFP at time n (time .)

IOU: σ^* = minimal optimal exercise time

IOU: Replication.

Obs 1: The process $\boxed{D_n V_n}$ is a super-martingale under $\tilde{P}$

M_n is a martingale (under $\tilde{P}$) if $\tilde{E}_n M_{n+1} = M_n$

Def: ⓐ We say an adapted process M_n is a super-mg (under $\tilde{P}$)

if $\tilde{E}_n M_{n+1} \le M_n$.

ⓑ M is a sub-mg if $\tilde{E}_n M_{n+1} \ge M_n$.

Pf of Obs 1: NTS $D_n V_n$ is a super mg under $\tilde{P}$.

i.e. NTS $\tilde{E}_n(D_{n+1} V_{n+1}) \le D_n V_n$

$\Leftrightarrow$ NTS $\quad V_n \geq \dfrac{\tilde{E}_n(D_{n+1}V_{n+1})}{D_n}$.

This is true because $V_n \stackrel{\text{def}}{=} \max\left\{G_n, \frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1})\right\}$.

$\Rightarrow D_n V_n$ is a super mg under $\tilde{P}$.

Thm: (Doob) $\underbrace{\text{(IOU Pf)}}$. Any super mg can be written as a martingale — (a predictable inc process).

Obs 2: Apply Doob to $D_n V_n$.

$\Rightarrow \exists$ a predictable $\boxed{\text{inc}}$ process A & a mg M $\ni$

$$D_n V_n = M_n - A_n \qquad (\text{Choose } \boxed{A_0 = 0})$$

Set $X_n = \dfrac{M_n}{D_n} \rightarrow D_n V_n = D_n X_n - \underbrace{A_n}$

(On HW today → Explicit formula for A)

③ $\Rightarrow X_n =$ wealth of some self fin portfolio

Claim: ① $X_n \geq G_n$ & ② On $\{n < \tau^*\}$, $X_n = V_n > G_n$

② $X_{\tau^*} = G_{\tau^*}$

Note ① + ② $\Rightarrow$ X is the wealth of the rep port of american option.
(& self fin) (& τ^* is an optimal exercise time).

③ $\Rightarrow$ τ^* is the minimal optimal exercise time.

($\because$ Say τ is any stopping time + $\tau \leq \tau^*$
& $P(\tau < \tau^*) > 0$

Then $\tilde{E}(D_\tau X_\tau)$ $\quad$ $E(D_{\tau}^{*} G_{\tau}^{*})$

will return to this shortly.

↳ Pf of claim: ① $X_n \geq G_n$.

Note $D_n V_n = D_n X_n - A_n \Leftrightarrow X_n = V_n + \frac{A_n}{D_n}$

Note A_n is increasing & $A_0 = 0 \Rightarrow A_n \geq 0$

$\Rightarrow X_n \geq V_n = \max\{G_n, \text{———}\}$

$\Rightarrow X_n \geq G_n$. Q.E.D.

Claim ②: $X_{\tau^*} = G_{\tau^*}$

$\tau^* = \min\{n \mid V_n = G_n\}$.

$$\boxed{\underline{\text{Claim}}: \forall n \le \tau^*, \text{ we must have } A_n = 0}$$

$(\Rightarrow X_n = V_n \text{ whenever } n \le \tau^*)$

$$\Rightarrow X_{\tau^*} = V_{\tau^*} = G_{\tau^*}.$$

IOU.

$\hookrightarrow$ Let's prove this: note $V_n = \max\left\{ \frac{1}{D_n} \tilde{E}_n(D_{n+1} V_{n+1}),\ G_n \right\}$,

$\Rightarrow$ whenever $n \le \tau^*$, we must have $V_n = \frac{1}{D_n} \tilde{E}_n(D_{n+1} V_{n+1})$

$\Rightarrow$ when $\quad n \lneq \tau^*, \quad D_n V_n = \tilde{E}_n \left(D_{n+1} V_{n+1} \right)$

Note $\quad D_n X_n = \tilde{E}_n \left(D_{n+1} X_{n+1} \right) \quad \forall n.$

& $\quad D_n V_n \geq D_n X_n \, \cancel{\geq} \, \underline{=} \, A_n$

$$\mathbb{1}_{\{n \lneq \tau^*\}} \tilde{E}_n \left(D_{n+1} V_{n+1} \right) = \tilde{E}_n \left(\mathbb{1}_{\{n \lneq \tau^*\}} D_{n+1} V_{n+1} \right) =$$

$\|$

$$\mathbb{1}_{\{n \lneq \tau^*\}} D_n V_n .$$

$$\mathbb{1}_{\{n \lneq \tau^*\}} \tilde{E}_n\left(D_{n+1} X_{n+1} - \underbrace{A_{n+1}}\right)$$

$$(\because A \text{ is pred})$$

$$\|$$

$$\mathbb{1}_{\{n \lneq \tau^*\}}\left(D_n X_n - A_{n+1}\right) = \mathbb{1}_{\{n \lneq \tau^*\}} D_n V_n$$

$$\Rightarrow \mathbb{1}_{\{n \lneq \tau^*\}}\left(\underbrace{D_n X_n - A_n}_{D_n V_n} + A_n - A_{n+1}\right) = \mathbb{1}_{\{n \lneq \tau^*\}} D_n V_n$$

$$\Rightarrow \mathbb{1}_{\{n \leq \tau^*\}} (A_{n+1} - A_n) = 0$$

$$\boxed{\Rightarrow A_n = 0 \quad \forall n \leq \tau^*.}$$

(Will finish the rest next time)