Last time: American options $\to$ intrinsic value $G = (G_0, G_1, \dots G_N)$

$\to$ can exercise at any stopping time

① V_0^{τ} = AFP of an option with maturity time τ & payoff G^{τ}
(at time 0) (τ = finite stopping time)

② Sell to highest bidder $\Rightarrow$ AFP of American opt $= \max_{\tau} V_0^{\tau}$

③ Replication: To replicate an american option we need to find
a self fin portfolio (wealth X_n) such that

(1) $X_n \geq G_n \quad \forall n \qquad (\Leftrightarrow X_\tau \geq G_\tau \ \forall$ finite stopping times $\tau)$

& (2) $X_{\tau^*} = G_{\tau^*}$ for at least one finite ~~stopping time~~

called ~~the~~ a optimal exercise time

last time: $X_0 = V_0^{\tau^*} = \max_\tau V_0^\tau$

Question 6.60. *Is the wealth of the replicating portfolio (for an American option) uniquely determined?*

Say X & Y are the wealth processes of ~~the~~ R. port of American option.

Know: ① $X_n \geq G_n$ & $Y_n \geq G_n$.

& ② $X_{\tau^*} = G_{\tau^*}$ for any finite stopping time τ^* $\quad$ $Y_{\tau^*} = G_{\tau^*}$ for 1 finite stopping time τ^*

<u>Claim:</u> In an AF market, $X_{\tau^*} = Y_{\tau^*}$ $\left(\& \; X_{\tau^*} = Y_{\tau^*} \right)$

Pf: Know $X_{\tau^*} \geq G_{\tau^*} = Y_{\tau^*}$.

Also, AFP at time 0 $= \tilde{E}\left(D_{\tau^*} \; G_{\tau^*}\right) = V_0 = Y_0$

$V_0 = X_0 \overset{OST}{=} \tilde{E}\left(D_{\tau^*} X_{\tau^*}\right)$ $\quad \tilde{E}\left(D_{\tau^*} G_{\tau^*}\right)$

Note $D_{\tau^*} X_{\tau^*} \geq D_{\tau^*} G_{\tau^*}$

$\Rightarrow X_{\tau^*} = G_{\tau^*}$

QED.

Claim (IOU): Before the optimal exercise time

$X_n = Y_n$

If $EZ = EW$

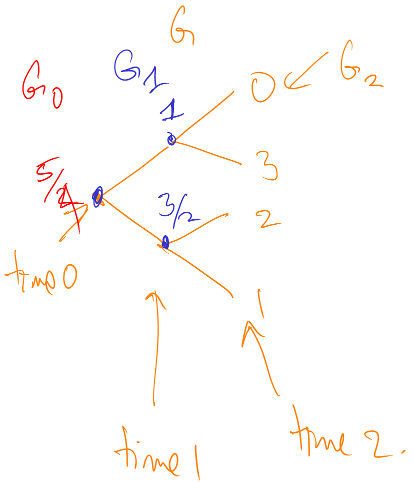
$\overset{?}{\Rightarrow} Z = W$

But

$Z \geq W$ & $EZ = EW$

$\Rightarrow Z = W$

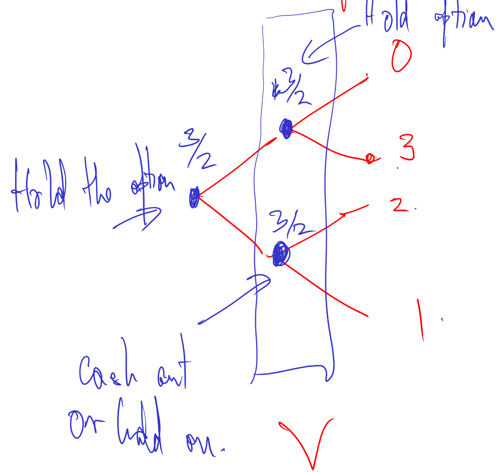
Question 6.61. *How do you find the minimal optimal exercise time, and the arbitrage free price? Let's take a simple example first.*



made up #'s for ex

Choose for simplicity $r = 0$, $\tilde{p} = \tilde{q} = \frac{1}{2}$.

AFP of this option is



AFP at time 2

AFP at time 1

$G = (G_0, G_1, \dots, G_N)$

Theorem 6.62. *Consider an American option with intrinsic value G. Define*

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time.

Remark 6.63. For the binomial model with $0 < d < 1 + r < u$ the above simplifies to

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \dots, \omega_n)\,.$$

To prove Thm 6.62: IOU → ~~Need~~

① Need to find a self financing portfolio such with wealth process X_n such that: ⓐ $X_n \geq G_n \ \forall n$

ⓑ $X_{\sigma^*} = G_{\sigma^*}$

moreover ⓒ whenever $n < \sigma^*$, $X_n = V_n > G_n$