6.5. **American Options.** An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.51. Let $G_0, G_1, \ldots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.52. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.53. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*

Strategy I: Let σ be a finite stopping time, and consider an option with (random) maturity time σ and payoff G_σ. Let V_0^σ denote the arbitrage free price of this option. The arbitrage free price of the American option *should be* $V_0 = \max_\sigma V_0^\sigma$, where the maximum is taken over all *finite* stopping times σ.

Definition 6.54. The *optimal exercise time* is a stopping time σ^* that maximizes $V_0^{\sigma^*}$ over all finite stopping times.

$\to V_0^{\sigma^*} = \max_\sigma = V_0^\sigma$

Definition 6.55. An optimal exercise time σ^* is called *minimal* if for every optimal exercise time τ^* we have $\sigma^* \leqslant \tau^*$.

Remark 6.56. The optimal exercise time need not be unique. (The *minimal* optimal exercise time is certainly unique.)

Say have 1 american option.

Pick σ → finite stopping time & sell this option with fixed maturity $\boxed{\sigma}$

Know how to price V_0^σ = price

Sell to highest bidder → Can sell for $\max_\sigma V_0^\sigma$

max over all finite stopping times τ

Guess: P^{AF}_{rice} of american option $= \underline{\max_{\tau} V_0^{\tau}} = V_0^{\tau^*}$

$\Vert$

V_0

Question 6.57. *Does this replicate an American option? Say σ^* is the optimal exercise time, and we create a replicating portfolio (with wealth process X) for the option with payoff G_{σ^*} at time σ^*. Suppose an investor cashes out the American option at time τ. Can we pay him?*

$$V_0^{\sigma^*} = \max_{\sigma} V_0^{\sigma}$$

X_n = wealth at time n of the portfolio required to replicate the option with fixed maturity σ^* & payoff G_{σ^*}.

Q: Does this "replicate" the American option?

(No → If he exercises at a different stopping time τ, not guaranteed we can pay him)!

Strategy II: Replication. Suppose we have sold an American option with intrinsic value G to an investor. Using that, we hedge our position by investing in the market/bank, and let X_n be the our wealth at time n.

(1) Need $X_\sigma \geqslant G_\sigma$ for all finite stopping times σ. (Or equivalently $X_n \geqslant G_n$ for all n.)

(2) For (at-least) *one* stopping time σ^*, need $X_{\sigma^*} = G_{\sigma^*}$.

The arbitrage free price of this option is X_0.

Say the american option can be traded for $X_0 + \varepsilon$ at time 0 $(\varepsilon > 0)$

↳ at time 0, sell option for $X_0 + \varepsilon$
buy Rep strat for X_0, ε in bank

→ Wealth at any time n
$\varepsilon(1+r)^n + X_n - G_n$

or at any stopping time τ, Wealth $= \underbrace{\varepsilon(1+r)^\tau}_{\text{profit}} + \underbrace{X_\tau - G_\tau}_{\geqslant 0 \text{ (no risk)}}$ ⇒ arbitrage

Say we can trade at the american option at price $X_0 - \varepsilon$ at time 0

↳ Then be at time $0 \to$ buy option for $X_0 - \varepsilon$

short rep portfolio for X_0

ε in bank.

My Wealth at time $n = G_n - X_n + \varepsilon(1+r)^n$

Wealth at any stopping time $\tau = G_\tau - X_\tau + \varepsilon(1+r)^\tau$

Choose $\tau = \tau^* =$ optimal exercise time $\Rightarrow$ Wealth $= \underbrace{G_{\tau^*} - X_{\tau^*}}_{0} + \underbrace{\varepsilon(1+r)^\tau}_{\text{profit.}} \Rightarrow$ arb!

Proposition 6.58. *In the binomial model with* $\boxed{0 < d < 1 + r < u,}$ *we must have* $\boxed{X_0 = \max\{V_0^\sigma \mid \sigma \text{ is a finite stopping time }\}.}$

Remark 6.59. The above is true in any complete, arbitrage free market.

X_n = wealth at time n of R. Portfolio above.

Pf: ① Check $X_0 \geq V_0^\tau$ $\forall$ finite stopping times τ.

Pf: For fixed exercise time τ,

AFP at time n (when $n \leq \tau$) $= \frac{1}{D_n} \tilde{E}_n (D_\tau G_\tau) \mathbb{1}_{\{n \leq \tau\}}$

Put $n=0$: $V_0^\tau = \tilde{E}(D_\tau G_\tau) \leq \tilde{E}(D_\tau X_\tau) \overset{OST}{=} \tilde{E}(D_0 X_0) = X_0$

① R. Portfolio

mg under $\tilde{P}$

$$\Rightarrow X_0 \geq V_0^{\tau} \quad \forall \tau$$

$$\Rightarrow X_0 \geq \max\{V_0^{\tau} \mid \tau \text{ is any finite stopping time}\}.$$

② Claim: $X_0 \leq \max\{V_0^{\tau} \mid \tau \text{ is any finite stopping time}\}$.

Pf: Choose $\tau = \tau^* =$ optimal exercise time.

QED.

Knows $X_{\tau^*} = G_{\tau^*}$.

$$\& \quad V_0^{\tau^*} = \tilde{E}(D_{\tau^*} G_{\tau^*}) = \tilde{E}(D_{\tau^*} X_{\tau^*}) \overset{OST}{=} \tilde{E} D_0 X_0 = X_0$$

Question 6.60. *Is the wealth of the replicating portfolio (for an American option) uniquely determined?*