Question 6.46. *Say M is a martingale. We know $\boldsymbol{E}M_n = \boldsymbol{E}M_0$ for all n. Is this also true for stopping times?*

$$\boldsymbol{E}M_0 = M_0$$

Q: $E M_\tau = E M_0$ for some stopping time τ

Theorem 6.47 ($\boxed{\text{Doob's optional sampling theorem}}$). *Let τ be a bounded stopping time and M be a martingale. Then $\boldsymbol{E}_n M_\tau = M_{\tau \wedge n}$.*

OST.

Remark 1: Know $E_n M_{n+1} = M_n$.

Choose $\tau = n+1$ (stopping time)

OST: $E_n M_\tau = M_{\tau \wedge n}$

$E_n M_{n+1}$

$M_{n \wedge (n+1)} = M_n$. ✓

Proof of OST: τ is bdd $\left(\Rightarrow P(\tau = \infty) = 0 \right)$

$$E_n M_\tau = E_n \left(\sum_{k=0}^{N} \mathbb{1}_{\{\tau = k\}} M_k \right) = \sum_{k=0}^{N} E_n \mathbb{1}_{\{\tau = k\}} M_k$$

$$= \sum_{k=0}^{n} E_n \mathbb{1}_{\{\tau = k\}} M_k + \sum_{k=n+1}^{N} E_n \mathbb{1}_{\{\tau = k\}} M_k$$

Can remove E_n ($\because \tau$ a stopping time $\Rightarrow \{\tau = k\} \in \mathcal{F}_k \ \forall k$)

$$= \sum_{k=0}^{n} \mathbb{1}_{\{\tau = k\}} M_k + \sum_{k=n+1}^{N} E_n \mathbb{1}_{\{\tau = k\}} E_k M_N$$

($\because M$ is a mg)

$$= \sum_{k=0}^{N} \mathbb{1}_{\{\tau=k\}} M_k + \sum_{k=n+1}^{N} \underbrace{E_n E_k}_{E_n} \left(\mathbb{1}_{\{\tau=k\}} M_N \right)$$

$$= \quad '' \quad + \sum_{k=n+1}^{N} E_n \left(\mathbb{1}_{\{\tau=k\}} M_N \right)$$

$$= \quad '' \quad + E_n \left(\underbrace{\left(\sum_{k=n+1}^{N} \mathbb{1}_{\{\tau=k\}} \right)}_{} M_N \right)$$

$$= \quad '' \quad + E_n \left(\mathbb{1}_{\{\tau>n\}} M_N \right)$$

$$\left(\because \{\tau > n\} = \{\tau \le n\}^c \in \mathcal{F}_n \right)$$

$$= \sum_{k=0}^{n} \mathbb{1}_{\{\tau=k\}} M_{\cancel{k}\,\tau} + \mathbb{1}_{\{\tau>n\}} \underbrace{E_n M_N}$$

$$= \mathbb{1}_{\{\tau\le n\}} M_{\tau\wedge n} + \mathbb{1}_{\{\tau>n\}} M_{n\wedge\tau}$$

$$= \left(\mathbb{1}_{\{\tau\le n\}} + \mathbb{1}_{\{\tau>n\}}\right) M_{n\wedge\tau} = M_{n\wedge\tau}$$

QED.

Proposition 6.48. *Suppose a market admits a* $\boxed{\text{risk neutral measure}}$. *If X is the wealth of a self-financing portfolio and τ is a finite stopping time such that* $\boxed{X_0 = 0,}$ *and $X_\tau \geqslant 0$, then $X_\tau = 0$.* (a.s)

Remark 6.49. This is simply an alternate proof of Proposition 6.45.

$\tilde{E}(D_n X_n)$ is a mg whenever X_n is the wealth of a self financing Pf.

Know $D_n X_n$ is a mg under $\tilde{P}$ (RNM).

$$\longrightarrow E(D_\tau X_\tau) = E_0(D_\tau X_\tau) \overset{\text{OST}}{=} D_0 X_0 = 0$$

Note $X_\tau \geq 0$ a.s. $\Rightarrow D_\tau X_\tau \geq 0$ a.s. $(\because D_\tau > 0)$

Since $D_\tau X_\tau \geq 0$ & $E D_\tau X_\tau = 0 \Longrightarrow D_\tau X_\tau = 0$ (a.s.)

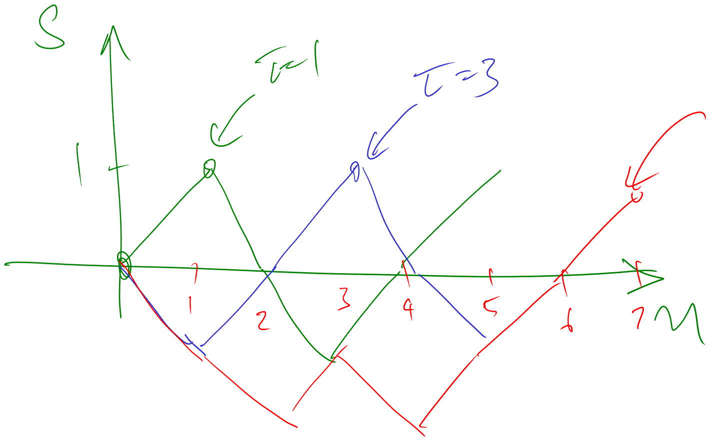
$\Rightarrow X_\tau = 0$ a.s. QED.

Remark: OST $\Rightarrow$ If M is a mg & τ is a $\boxed{\text{bdd}}$ stopping time $\Big\}$ $\Rightarrow$ $EM_\tau = EM_0$

$$\left(\because EM_\tau = E E_0 M_\tau \overset{OST}{=} EM_0 \right).$$

Question 6.50 (Gamblers ruin). *Suppose $N = \infty$. Let X_n be i.i.d. random variables with mean 0, and let $S_n = \sum_1^n X_k$. Let* $\tau = \min\{n \mid S_n = 1\}$. *(It is known that $\tau < \infty$ almost surely.) What is $\boldsymbol{E} S_\tau$? What is* $\lim_{N\to\infty} \boldsymbol{E} S_{\tau \wedge N}$? $= 0$

$S_0 = 0, \quad S_1 = X_1, \quad S_2 = X_1 + X_2 \quad --$



Note ① τ is a stopping time

② S_n is a mg

$\left(\because E_n S_{n+1} = E_n(S_n + X_{n+1})\right.$

$\left. = S_n + E X_{n+1} = S_n \right)$

③ $E S_\tau = E 1 \neq \underbrace{E S_n}_{0} \; \forall n$

④ Claim: $\tau < \infty$ almost surely (τ is a finite stopping time)

⑤ Does this contradict OST (No because τ is NOT bdd)

Does this give you an arb of at gambling? ← NO

(because you can run out of money before winning $1)