Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$X_N \mathbf{1}_{\{\sigma = N\}} = G_N \mathbf{1}_{\{\sigma = N\}}$$

$$X_n \mathbf{1}_{\{\sigma \geqslant n\}} = G_n \mathbf{1}_{\{\sigma = n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma > n\}} \tilde{\boldsymbol{E}}_n X_{n+1} \, .$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1) \, .$$

Last time: $\sigma \longrightarrow$ stopping time (finite)

Security pays G_σ at the random time σ.

As before, we will use state processes to find practical algorithms to price securities.

Proposition 6.43. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let $A_1, \dots, A_N \subseteq \mathbb{R}^d$, with $A_N = \mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\}\,.$$

Let $g_0, \dots g_N$ be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n)\mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation then

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r}\Big(\tilde{p} f_{n+1}(h_{n+1}(y,1)) + \tilde{q} f_{n+1}(h_{n+1}(y,-1))\Big)$$

$\leftarrow y \in \text{Range}(Y_n)$

Eg: Up rebate option $\to$ pays first time ~~when~~ $S_n \geq U$.

$d=2 \to$ Set $Y_n = (S_n, M_n)$ $\qquad (M_n = \max\{S_0, S_1, S_2 \cdots S_n\})$

$A_n = \mathbb{R} \times [U, \infty) = \{(s,m) \mid m \geq U\}$. $\quad (Y_n \in A_n \Rightarrow M_n \geq U \Rightarrow$ sec pays$)$

Proof: ① At time N: $\quad AFP = X_N \mathbb{1}_{\{\tau = N\}} = G_N \mathbb{1}_{\{\tau = N\}}$

$$= \underline{g_N(Y_N)} \mathbb{1}_{\{\tau = N\}}$$

$$\Rightarrow f_N(y) = g_N(y).$$

② Ind step: Compute f_n:

AFP at time n $= X_n \mathbb{1}_{\{\tau > n\}} = G_n \underline{\mathbb{1}_{\{\tau = n\}}} + \left(\tilde{E}_n \mathbb{1}_{\{\tau > n\}} X_{n+1}\right) \frac{1}{1+r}$

$$= g_n(Y_n) \mathbb{1}_{\{Y_n \in A_n\}} + \tilde{E}_n \mathbb{1}_{\{Y_n \notin A_n\}} f_{n+1}(Y_{n+1}) \frac{1}{1+r}$$

$$Y_{n+1} = h_{n+1}(Y_n, \omega_{n+1})$$

$$= g_n(Y_n)\mathbb{1}_{\{Y_n \in A_n\}} + \frac{1}{1+r}\tilde{E}_n \mathbb{1}_{\{Y_n \notin A_n\}} f_{n+1}\left(h_{n+1}\left(\underline{Y_n}, \underline{\omega_{n+1}}\right)\right)$$

indep lemma

$$= g_n(Y_n)\mathbb{1}_{\{Y_n \in A_n\}} + \frac{1}{1+r}\mathbb{1}_{\{Y_n \notin A_n\}}\left(\tilde{p}\, f_{n+1}(h_{n+1}(Y_n, 1)) + \tilde{q}\, f_{n+1}(h_{n+1}(Y_n, -1))\right).$$

fn of Y_n. Set $Y_n = y$ & get nec rel.

QED.

6.4. **Optional Sampling.** Consider a market with a few risky assets and a bank. (interest rate r)

Question 6.44. *If there is no arbitrage opportunity at time N, can there be arbitrage opportunities at time $n \leqslant N$? How about at finite stopping times?*

$X_0 = 0$, X = wealth of a self fin portfolio.

$$X_N \geq 0 \implies X_N = 0 \text{ a.s.}$$

Can $\exists\, n \leq N$ + $X_0 = 0$, $X_n \geq 0 \overset{?}{\implies} X_n = 0$

Yes: Must have $X_n = 0$. (Otherwise $\to$ transfer all \$ to bank at time n & get an arb oppty at time N).

Can $\exists$ a finite stopping time τ $\ni$ $X_0 = 0$, $X_{\tau} \geq 0$ & $X_\tau \neq 0$ identically?

Claim NO. If such a τ exists,

transfer wealth to bank at time τ

& get an arb opp at time N. (some details to be checked)

Proposition 6.45. *There is no arbitrage opportunity at time N if and only if there is no arbitrage opportunity at any finite stopping time.*

(You check)

Question 6.46. *Say M is a martingale. We know $\boldsymbol{E}M_n = \boldsymbol{E}M_0$ for all n. Is this also true for stopping times?*

$$\left(\because \quad EM_{n+1} = E\,E_n M_{n+1} = EM_n \right)$$

Is $EM_\tau = EM_0$ for finite stopping times τ ?

$$X_n \begin{cases} +1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}, \quad X_n \text{ iid.}$$

τ = first time $M_n = \underline{1000}$

$$M_n = \sum_1^n X_k$$

Q: $EM_n = 0$.
M is a mg.

Q2: $\underline{EM_\tau} = \underline{1000}$

Theorem 6.47 (Doob's optional sampling theorem)**.** *Let τ be a bounded stopping time and M be a martingale. Then* $\boxed{\boldsymbol{E}_n M_\tau} = M_{\tau \wedge n}$.

Note OST $\Rightarrow$ $E M_\tau = E_0 M_\tau \overset{OST}{=} M_{\tau \wedge 0} = M_0$

OST $\Rightarrow$ $E M_\tau = M_0$ (not random)

$= EM_0$.

ne

$Y = (Y^1 \dots Y^d)$ $\quad d \in \{1, 2, \dots\}$

State process: Security pays $\underline{g_N(Y_N)}$ at time N.

(1) $\Longrightarrow$ AFP at time n $= \underline{g_n(Y_n)}$ for some fn g_n.

Thm (1): $Y_{n+1} = h_{n+1}(Y_n, \underline{\omega_{n+1}}) \Longrightarrow Y$ is a state process.

Thm (2) Y is markov (& int rate not random) $\Longrightarrow$?