1. $\hat{\omega} = (\hat{\omega}_1, \dots, \hat{\omega}_N) \in \Omega$.

$X \in \mathcal{F}_n$, $\underline{X = x_1, \dots, x_m}$.

$$\cancel{\mathbb{E}_n[X(\omega)] = \frac{\sum_{\omega' \in \Pi_n(\omega)} p(\omega')X(\omega')}{\sum_{\omega' \in \Pi_n(\omega)} p(\omega')}, \quad \Pi_n(\omega) = \{\omega' \mid \omega_1' = \omega_1, \dots, \omega_n' = \omega_n\}}$$

① $\sum_{r=1}^{m} x_r \, \mathbb{P}[X = x_r \mid \underbrace{\omega_1 = \hat{\omega}_1, \dots \omega_n = \hat{\omega}_n}_{\in \mathcal{F}_n}] \in \mathcal{F}_n$

② $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} \mathbb{E}_n(X(\omega)) \, p(\omega) = \sum_{\omega \in A} X(\omega) \, p(\omega)$

$$= \sum_{r=1}^{m} x_r \, \mathbb{P}(X = x_r \cap A).$$

$$= \sum_{r=1}^{m} x_r \sum_{\omega \in \{X = x_r\} \cap A} p(\omega).$$

$$\mathbb{E}_n(X(\hat{\omega})) = \mathbb{E}_n(X(\hat{\omega}_1, \dots \hat{\omega}_N)) = \mathbb{E}_n(X(\hat{\omega}_1, \dots, \hat{\omega}_n)).$$

Let $\underline{A = \{\omega \mid \omega_1 = \hat{\omega}_1, \dots, \omega_n = \hat{\omega}_n\}}$.

$$\sum_{\omega \in A} \mathbb{E}_n(X(\omega)) \, p(\omega) = \sum_{\omega \in A} \mathbb{E}_n(X(\hat{\omega})) \, p(\omega) = \sum_{r=1}^{m} x_r \, \mathbb{P}(\{x_r = X\} \cap A)$$

$$\mathbb{E}_n(X(\hat{\omega}_1, \dots, \hat{\omega}_n)) = \mathbb{E}_n(X(\hat{\omega}))$$

$$\mathbb{E}_n(X(\hat{\omega})) \, P(A) = \sum_{r=1}^{m} x_r \, \mathbb{P}(\{x_r = X\} \cap A)$$

$$\mathbb{E}_n(X(\hat{\omega})) = \sum_{r=1}^{m} x_r \, \mathbb{P}[X = x_r \mid A] = \sum_{r=1}^{m} x_r \, \mathbb{P}[X = x_r \mid \omega_1 = \hat{\omega}_1, \dots, \omega_n = \hat{\omega}_n]$$

3.

(a) $Y_n = \sum_{k=0}^{n} S_k$ need not be Markov process.

$$\tilde{E}_n[f(Y_{n+1})] \overset{?}{=} \underline{g(Y_n)}.$$

$$Y_{n+1} = \underbrace{\sum_{k=0}^{n} S_k}_{\in F_n} + S_{n+1}, \quad \tilde{E}_n[f(Y_{n+1})]$$

$$f(Y_n, S_{n+1}) = f(Y_n, uS_n)\tilde{p} + f(Y_n, dS_n)\tilde{q}.$$

$g(Y_n)$ no information on S_n specifically, so Y_n is not necessarily Markov process.

(b) $X_n = (S_n, Y_n)$ is Markov Process.

$$\tilde{E}_n[f(X_{n+1})] = \tilde{E}_n[f(S_{n+1}, Y_{n+1})] = \tilde{E}_n[f'(S_{n+1}, Y_n)]$$

$\frac{S_{n+1}}{S_n}$ $\quad Y_{n+1} - S_n$

$$= \tilde{p} \cdot f'(uS_n, Y_n) + \tilde{q} f'(dS_n, Y_n) \quad (\text{by HW3. a3})$$

$$= g(S_n, Y_n)$$

$$\tilde{E}_n\left(f'\left(\frac{S_{n+1}}{S_n}, Y_n\right)\right) = \tilde{p} f'(u, Y_n) + \tilde{q} f'(d, Y_n)$$

$$= \tilde{p} f(uS_n, Y_n + uS_n) + \tilde{q}$$

$$\underline{f(S_{n+1}, Y_{n+1}) = f'\left(\frac{S_{n+1}}{S_n}, Y_n\right)}$$

indep. F_n $\quad \in F_n$

(c) ① $f_N(S_N, Y_N) = f(Y_N)$

$f_n(S_n, Y_n)$ is arbitrage free price at time n, which means

$\frac{f_n(S_n, Y_n)}{(1+r)^n}$ is Martingale w.r.t. $\tilde{P}$.

$$\tilde{E}_n\left(f_{n+1}(S_{n+1}, Y_{n+1}) / (1+r)^{n+1}\right) = f_n(S_n, Y_n)/(1+r)^n$$

$$\tilde{E}_n\left(f_{n+1}(S_{n+1}, Y_{n+1})\right) = \boxed{(1+r) f_n(S_n, Y_n)}$$

$$\tilde{E}_n\left(f'_{n+1}\left(\frac{S_{n+1}}{S_n}, Y_n\right)\right) = \tilde{p} f'_{n+1}(u, Y_n) + \tilde{q} f'_{n+1}(d, Y_n)$$

$$= \boxed{\tilde{p} f_{n+1}(uS_n, Y_n + uS_n) + \tilde{q} f_{n+1}(dS_n, Y_n + dS_n)}$$

② $(1+r) f_n(S_n, Y_n) = \tilde{p} f_{n+1}(uS_n, Y_n + uS_n) + \tilde{q} f_{n+1}(dS_n, Y_n + dS_n)$

for $n = 0, \dots, N-1$.

$\cancel{f'(a,b) = f(a, a+b)}$ $\quad f(S_n, Y_n) = f'\left(\frac{S_n}{S_{n-1}}, Y_n - S_n\right).$

X independent from F_n, Y is F_n-measurable,

$$E_n[f(\underset{S_{n+1}}{X}, \underset{Y_n}{Y})] = \sum_{i=1}^{m} f(x_i, Y) P[X = x_i].$$

4.

$$V_N = \frac{1}{N}\sum_{n=1}^{N}\left(\log\left(\frac{S_n}{S_{n-1}}\right)\right)^2 - k^2 .$$

$V_0 = 0.$ $\qquad V_n = \frac{1}{D_n}\tilde{\mathbb{E}}_n(D_N V_N)$

$$\underline{V_n D_n} = \underline{\tilde{\mathbb{E}}_n(D_N V_N)}.$$

$V_n D_n = \frac{V_n}{(1+r)^n}$ is MRT since arbitrage free. $\quad V_0 D_0 = 0 \Rightarrow V_n D_n$ is MRT w.r.t. $\tilde{\mathbb{E}}, \tilde{\mathbb{P}}$

$$\tilde{\mathbb{E}}\left[\frac{V_N}{(1+r)^N}\right] = 0 = \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}_{N-1}\left[\frac{V_N}{(1+r)^N}\right]\right]$$

$$\tilde{\mathbb{E}}_{N-1}\left[\frac{V_N}{(1+r)^N}\right] = \frac{1}{(1+r)^N}\tilde{\mathbb{E}}_{N-1}\left[\frac{1}{N}\left(\sum_{n=1}^{N-1}\left(\log\left(\frac{S_n}{S_{n-1}}\right)\right)^2 + \left(\log\left(\frac{S_N}{S_{N-1}}\right)\right)^2\right) - k^2\right]$$

$$= \frac{1}{(1+r)^N}\left[\frac{1}{N}\sum_{n=1}^{N-1}\left(\log\left(\frac{S_n}{S_{n-1}}\right)\right)^2 + \frac{1}{N}\tilde{\mathbb{E}}_{N-1}\left(\log\left(\frac{S_N}{S_{N-1}}\right)^2\right) - k^2\right]$$

$$= \frac{1}{(1+r)^N}\left[\underline{\frac{1}{N}\sum_{n=1}^{N-1}\left(\log\left(\frac{S_n}{S_{n-1}}\right)\right)^2} + \frac{1}{N}\left(\tilde{p}\,\overset{(\log(u))^2}{\cancel{\log(u^2)}} + \tilde{q}\,\overset{(\log(d))^2}{\cancel{\log(d^2)}}\right) - k^2\right]$$

$$\tilde{\mathbb{E}}_{N-2}\left[\frac{1}{N}\log\left(\frac{S_{N-1}}{S_{N-2}}\right)\right)^2\right]$$

$$\tilde{\mathbb{E}}_{N-2} \rightarrow \frac{1}{(1+r)^N}\left[\frac{1}{N}\sum_{n=1}^{N-2}\left(\log\left(\frac{S_n}{S_{n-1}}\right)\right)^2 + \frac{2}{N}\left(\tilde{p}\log^2(u) + \tilde{q}(\log(d))^2\right). - k^2\right]$$

$$\boxed{\tilde{\mathbb{E}}_n\left[\left(\log\left(\frac{S_{n+1}}{S_n}\right)\right)^2\right] = \tilde{p}(\log(u))^2 + \tilde{q}(\log(d))^2}$$

$$\frac{1}{(1+r)^N}\left(\frac{N}{N}\left(\tilde{p}(\log(u))^2 + \tilde{q}((\log(d))^2\right) - k^2\right) = 0.$$

$$k^2 = \tilde{p}(\log(u))^2 + \tilde{q}(\log(d))^2.$$