N time binomial model $\quad 0<d<1+r<u$

Variance Swap

$$V_N = \frac{1}{N}\sum_{n=1}^{N}\left(\log\left(\frac{S_n}{S_{n-1}}\right)\right)^2 - K^2$$

$$S_n = \begin{cases} uS_{n-1} \\ dS_{n-1} \end{cases} \qquad \frac{S_n}{S_{n-1}} = \begin{cases} u \\ d \end{cases}$$

$$\widetilde{\mathbb{E}}_0[V_N] = 0$$

$$K^2 = \widetilde{\mathbb{E}}_0\left[\frac{1}{N}\sum_{n=1}^{N}\left(\log\left(\frac{S_n}{S_{n-1}}\right)\right)^2\right]$$

$$\widetilde{\mathbb{E}}_0[Y] = \widetilde{\mathbb{E}}_0[\widetilde{\mathbb{E}}_1[\widetilde{\mathbb{E}}_2[\dots\widetilde{\mathbb{E}}_{N-1}[Y]\dots]$$

Asian option

$V_N = f\left(\sum_{n=0}^{N} \frac{S_n}{N}\right)$ $\quad f(Y_N)$ $\quad Y_N = \sum_{n=0}^{N} S_n$ $\quad g(a,b)$

$= \tilde{q} f(a+ub) + \dots$

$$\tilde{\mathbb{E}}_n[f(Y_N)] \overset{?}{=} f_n(Y_n, S_n)$$

$uS_n \quad dS_n$

$$(Y_{n+1}, S_{n+1}) = (Y_n + S_{n+1}, S_{n+1})$$

$$\tilde{\mathbb{E}}_{N-1}[f(Y_N)] = \tilde{\mathbb{E}}_{N-1}[f(Y_{N-1} + S_N)]$$

$S_N \begin{cases} uS_{N-1} \\ dS_{N-1} \end{cases}$ $\quad \tilde{p}$, $\tilde{q} = 1 - \tilde{p}$

$$= \tilde{p} f(Y_{N-1} + uS_{N-1}) + \tilde{q} f(Y_{N-1} + dS_{N-1})$$

$$Y_n = \sum_{i=0}^{n} S_i \qquad S_{n+1} = \begin{cases} uS_n \\ dS_n \end{cases}$$

$$Y_n - Y_{n-1} = S_n$$

$$\begin{aligned} Y_{n+1} &= Y_n + S_{n+1} \qquad \gamma \in (u,d) \\ &= Y_n + \gamma \underbrace{(Y_n - Y_{n-1})}_{S_n} \end{aligned}$$

$$Y_n = \sum_{i=0}^{n} S_i$$

$$f(Y_N)$$

$$\begin{cases} uS_{N-1} & \tilde{p} \\ dS_{N-1} & \tilde{q} \end{cases}$$

$$\tilde{\mathbb{E}}_{N-1}\left[f(Y_N)\right] = \tilde{\mathbb{E}}_{N-1}\left[f(Y_{N-1}+S_N)\right]$$

$$= \tilde{p}\, f(Y_{N-1}+uS_{N-1}) + \tilde{q}\, f(Y_{N-1}+dS_{N-1})$$