Last time: Securities with a random maturity time.

Payment time has to be a stopping time.

Recall: τ is a stopping time if

① $\tau : \Omega \longrightarrow \{0, \dots N\} \cup \{\infty\}$

& ② $\forall n \leq N$, $\underbrace{\{\tau \leq n\}}$ $\in$ $\underbrace{\mathcal{F}_n}$.

- Let G be an adapted process, and σ be a *finite* stopping time.
- Consider a derivative security that pays G_σ at the random time σ.
- Note $G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\sigma=n}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

$G_\sigma(\omega) = G_{\sigma(\omega)}(\omega)$

Definition 6.39. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$X_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Δ_n = # shares of stock held in the portfolio at time n.

$X_n = \frac{1}{D_n} \tilde{E}(D_\sigma G_\sigma \mathbb{1}_{\sigma \geq n})$

$\boxed{\text{when } n \leq \sigma}$

n

Pf: Let $X_n = \frac{1}{D_n} \tilde{E}_n(D_\tau G_\tau)$. Know X_n is $\mathcal{F}_n$-meas.

Claim: (1) $D_n X_n$ is a mg under $\tilde{P}$

Pf: $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n\left(\tilde{E}_{n+1}(D_\tau G_\tau)\right) \overset{\text{tower}}{=} \tilde{E}_n(D_\tau G_\tau)$

$= D_n X_n$ QED.

Note Claim 1 $\Longrightarrow$ X_n is the wealth of a self-financing portfolio.

Claim 2: $X_\tau = G_\tau$

Pf: $X_n = \frac{1}{D_n}\tilde{E}_n\left(D_\tau G_\tau\right)$

$$\Rightarrow X_n \cdot \mathbb{1}_{\{\tau=n\}} = \frac{1}{D_n}\left[\tilde{E}_n\left(D_\tau G_\tau\right)\right]\mathbb{1}_{\{\tau=n\}}$$

$$= \frac{1}{D_n}\tilde{E}_n\left(\mathbb{1}_{\{\tau=n\}} D_\tau G_\tau\right) \quad (\because \tau \text{ is a stopping time})$$

$$= \frac{1}{D_n}\tilde{E}_n\left(\mathbb{1}_{\{\tau=n\}} D_n G_n\right)$$

$$= \frac{1}{\cancel{D_n}}\left(\mathbb{1}_{\{\tau=n\}} \cancel{D_n} G_n\right) = \mathbb{1}_{\{\tau=n\}} G_\tau .$$

$$\Rightarrow X_\tau \mathbb{1}_{\{\tau=n\}} = G_\tau \mathbb{1}_{\{\tau=n\}} \qquad \boxed{\forall n}$$

$$\Rightarrow X_\tau = G_\tau \qquad \text{QED}$$

~~Cha~~ Hence : X is a replicating portfolio. ($\Rightarrow$ security can be replicated).

Compute AFP$_n$, when $n \le \tau$:

AFP when $\tau \geq n$ $= \underbrace{X_n \mathbb{1}_{\{\tau \geq n\}}} = \mathbb{1}_{\{\tau \geq n\}} \left(\frac{1}{D_n} \tilde{E}_n (D_\tau G_\tau) \right)$

$$= \underbrace{\frac{1}{D_n} \tilde{E}_n}\Big(\underbrace{D_\tau G_\tau \mathbb{1}_{\{\tau \geq n\}}}\Big)$$

(∵ τ is a stopping time

$\Rightarrow \{\tau \geq n\} \in \mathcal{F}_n$)

QED.

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$(1)\quad X_N \mathbf{1}_{\{\sigma=N\}} = G_N \mathbf{1}_{\{\sigma=N\}}$$

$$(2)\quad X_n \mathbf{1}_{\{\sigma\geqslant n\}} = G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r}\mathbf{1}_{\{\sigma>n\}} \tilde{\boldsymbol{E}}_n X_{n+1}\,.$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1)\,.$$

Pf: Know $X_n = \frac{1}{D_n}\tilde{E}_n(D_\tau G_\tau)$ $\Rightarrow$ $X_N \mathbb{1}_{\{\tau=N\}} = G_\tau \mathbb{1}_{\{\tau=N\}}$ $\Rightarrow$ (1)

Derive (2):

Note: $D_n X_n = \tilde{E}_{n\cancel{+1}}(D_{n+1} X_{n+1})$ $\quad$ ($\because$ $D_n X_n$ is a $\tilde{P}$ mg)

$\Rightarrow$ $X_n \mathbb{1}_{\{\tau \geq n\}} = \mathbb{1}_{\{\tau\geq n\}}\left(\frac{1}{D_n}\tilde{E}_{n\cancel{+1}}(D_{n+1}X_{n+1})\right)$

$$= \mathbb{1}_{\{\tau=n\}}\left(\frac{1}{D_n}\tilde{E}_n\left(D_{n+1}X_{n+1}\right)\right) + \mathbb{1}_{\{\tau>n\}}\left(\frac{1}{D_n}\tilde{E}_n\left(D_{n+1}X_{n+1}\right)\right)$$

$$= \mathbb{1}_{\{\tau=n\}}\frac{1}{\cancel{D_n}}\left(\cancel{D_n}X_n\right) + \frac{1}{D_n}\tilde{E}_n\left(\mathbb{1}_{\{\tau>n\}}\underline{D_{n+1}}X_{n+1}\right)$$

$$= \mathbb{1}_{\{\tau=n\}}G_\tau + \frac{1}{(1+r)}\tilde{E}_n\left(\mathbb{1}_{\{\tau>n\}}X_{n+1}\right)$$

QED