6.3. **Options with random maturity.** Consider the N period binomial model with $0 < d < 1 + r < u$.

Example 6.30 (Up-and-rebate option). Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Explicitly, let $\tau = \min\{n \leqslant N \mid S_n \geqslant U\}$, and let $\sigma = \tau \wedge N$. The up-and-rebate options pays $A\mathbf{1}_{\tau \leqslant N}$ at the random time σ.

inf

Remark 6.31. By convention $\min \emptyset = \infty$.

Up & rebate opt pays $A \mathbf{1}_{\{\tau \leq N\}}$ at time $\cancel{\tau}$ $\underbrace{\tau \wedge N}_{\sigma}$

$a \wedge b = \min\{a, b\}$

$a \vee b = \max\{a, b\}$

Definition 6.32. We say a random variable τ is a *stopping time* if:

(1) $\tau : \Omega \to \{0, \dots, N\} \cup \infty$

(2) For all $n \leqslant N$, the event $\{\tau \leqslant n\} \in \mathcal{F}_n$.

Remark 6.33. We say τ is a finite stopping time if $\tau < \infty$ almost surely.

Remark 6.34. The second condition above is equivalent to requiring $\{\tau = n\} \in \mathcal{F}_n$ for all n.

$$\boxed{\forall n \quad \{\tau \leq n\} \in \mathcal{F}_n} \iff \forall n \quad \{\tau = n\} \in \mathcal{F}_n$$

Question 6.35. *Is $\tau = 5$ a stopping time?*

Question 6.36. *Is the first time the stock price hits U a stopping time?*

Question 6.37. *Is the last time the stock price hits U a stopping time?*

$$\boxed{N = 370}$$

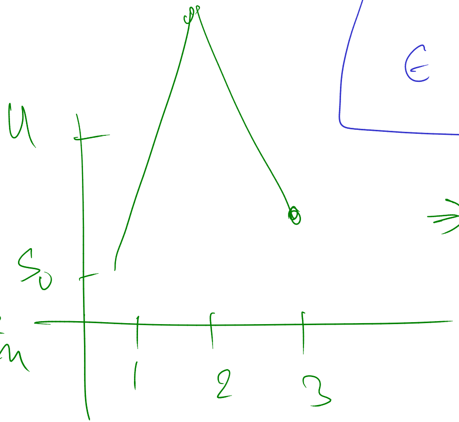
Yes: $\{\tau = n\} = \begin{cases} \Omega & n = 5 \\ \emptyset & n \neq 5 \end{cases} \in \mathcal{F}_0 \subseteq \mathcal{F}_n \ \forall n.$

$$\tau = \min\{n \mid S_n \geq U\}$$

$$\{\tau \leq n\} = \cancel{\{S_n \geq U\}}$$

$$= \{S_0 \geq U\} \cup \{S_1 \geq U\} \cup \dots \cup \{S_n \geq U\} \in \mathcal{F}_n$$

$$= \{\max\{S_0, S_1 \dots S_n\} \geq U\} \in \mathcal{F}_n$$



$\Rightarrow \{\tau \leq n\} \in \mathcal{F}_n \ \forall n$

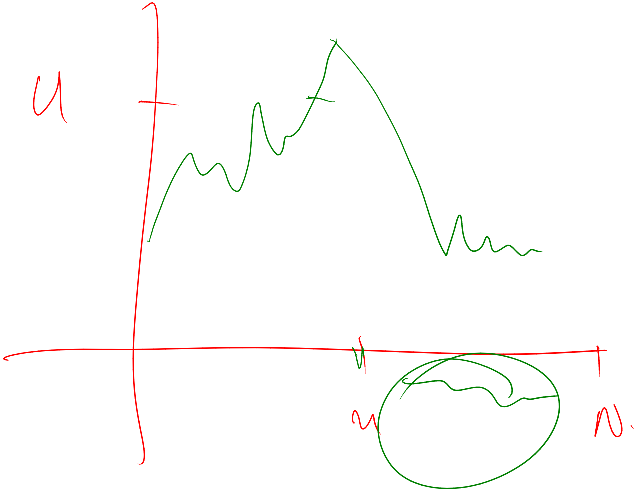
$\Rightarrow \tau$ is a stopping time.

Q: τ = last time S_n crosses U

NO! $\{\tau \le n\}$ involves knowing

$S_{n+1} \le U$, $S_{n+2} \le U$ ... etc.

$\notin \mathcal{F}_n$.



Question 6.38. *If σ and τ are stopping times, is $\sigma \wedge \tau$ a stopping time? How about $\sigma \vee \tau$?* Yes (use ~~union~~ intersection)

① $\sigma \wedge \tau : \Omega \longrightarrow \{0, \ldots N\} \cup \{\infty\}$ (Yes)

② NTS $\{\sigma \wedge \tau \le n\} \in \mathcal{F}_n \quad \forall n.$

Pf: $\{\sigma \wedge \tau \le n\} = \{\sigma \le n\} \cup \{\tau \le n\}$

$\{\omega \in \Omega \mid \sigma(\omega) \wedge \tau(\omega) \le n\}$

$\{\sigma \le n\} \in \mathcal{F}_n$ ($\because \sigma$ is a stopping time)

$\{\tau \le n\} \in \mathcal{F}_n$ ($\because \tau$ is a stopping time).

σ, τ is a stopping time

Is $\sigma/2$ a stopping time

$\{\sigma/2 \le n\} = \{\sigma \le 2n\} \in \mathcal{F}_{2n}$ need not be in $\mathcal{F}_n$.

- Let G be an adapted process, and σ be a *finite* stopping time.
- Consider a derivative security that pays G_σ at the random time σ.
- Note $\cancel{G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\sigma=n}}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

$$G_\sigma = \sum_{n=0}^{N} \mathbb{1}_{\{\sigma = n\}} G_n$$

Definition 6.39. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$\boxed{X_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})}$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

$$G_\sigma(\omega) = G_{\sigma(\omega)}(\omega)$$

X_0 = initial wealth

Δ_n = # shares of stock at time n held in the portfolio.

$(X_0, (\Delta_n))$ trad strat.

X_n = wealth at time n.

(RNP formula:
Payoff V_N at time N
the AFP at time n
$= \dfrac{1}{D_n} \tilde{E}_n (D_N V_N)$)