Last time: Markov Process: "Memoryless"

$$E_n\, g(X_{n+1}) = g(X_n)$$

$$\Leftrightarrow P(X_{n+1}=x_{n+1} \mid \boxed{X_1=x_1, X_2=x_2, \ldots} \; X_n=x_n)$$

$$= P(X_{n+1}=x_{n+1} \mid X_n=x_n)$$

Definition 7.11. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n.

Remark 7.12. For state processes given f_N, we find f_n by backward induction. The number of computations at time n is of order $\boxed{\text{Range}(Y_n)}$.

Remark 7.13. The fact that S_n is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process. (last time)

Y^i Super script $\longrightarrow$ coordinates of the process

Y^i_n Sub script $\longrightarrow$ time

$\quad$ || ith coordinate of Y at time n.

Notation for vector valued processes:

$$Y_n = (Y_n^1, Y_n^2, \ldots Y_n^d) = \begin{pmatrix} Y_n^1 \\ Y_n^2 \\ \vdots \\ Y_n^d \end{pmatrix}$$

Q: $E Y_n \stackrel{\text{def}}{=} (E Y_n^1, E Y_n^2, \ldots E Y_n^d) = \begin{pmatrix} E Y_n^1 \\ E Y_n^2 \\ \vdots \\ E Y_n^d \end{pmatrix}$

($Y_n^2 = 2^{\text{nd}}$ coordinate)

$(Y_n^2)^2$ = square.

Similarly $E_m Y_n$ = cond exp

$$= \begin{pmatrix} E_m Y_n^1 \\ \vdots \\ E_m Y_n^d \end{pmatrix}$$

(N period Binom, $0 < d < 1 + r < u$)

Theorem 7.14. *Let $Y = (Y^1, \ldots, Y^d)$ be a d-dimensional process. Suppose we can find functions $g_1, \ldots, g_N$ such that $Y_{n+1}(\omega) = g_{n+1}(Y_n(\omega), \omega_{n+1})$. Then Y is a state process.*

If $Y_{n+1}(\omega) = g_{n+1}\left(Y_n(\omega), \omega_{n+1}\right)$ $\quad g_{n+1} \to$ non random fn.

Pf: Let $m \leq N$. Consider a sec that pays $f_m(Y_m)$ at maturity m.

Let $n \leq m$. NTS: AFP at time $n = f_n(Y_n)$ for some fn f.

Pf: Backward induction. (1) True for $n = m$.

(2) Assume True for $n+1$ (i.e. assume AFP at time $n+1 = f_{n+1}(Y_{n+1})$)

Prove for n. (i.e. NTS AFP at time $n = f_n(Y_n)$ for some fn n.

Pf of (2): (RNP : Sec pays V_N at mat N

then AFP at time $n = \frac{1}{D_n} \tilde{E}_n D_N V_N$)

Consider a security that pays $f_m(Y_m)$ at time m.

$\Rightarrow$ AFP at time $n = \frac{1}{D_n} \tilde{E}_n \left(D_m f_m(Y_m) \right)$

$$\Rightarrow \text{AFP at time } n = \frac{1}{D_n} \left(\tilde{E}_n D_{n+1} \cdot \underbrace{(\text{AFP at time } n+1)}_{= f_{n+1}(Y_{n+1}) \text{ (ind Hyp)}} \right)$$

$\Rightarrow$ AFP at time n = $\dfrac{D_{n+1}}{D_n} \tilde{E}_n f_{n+1}(Y_{n+1}) = \dfrac{1}{1+r} \tilde{E}_n f_{n+1}(Y_{n+1})$

$\downarrow$ $/_{1+r}$

$$= \frac{1}{1+r} \tilde{E}_n f_{n+1}\Big(g_{n+1}\big(\underbrace{Y_n}_{\mathcal{F}_n\text{-meas}}, \underbrace{\omega_{n+1}}_{}\big)\Big)$$

indep of $\mathcal{F}_n$ under $\tilde{P}$

indep lma

$$= \underbrace{\frac{1}{1+r}\Big(f_{n+1}\circ g_{n+1}(Y_n, 1)\,\tilde{p} + f_{n+1}\circ g_{n+1}(Y_n, -1)\,\tilde{q}\Big)}_{\text{some non-random fn of } Y_n}$$

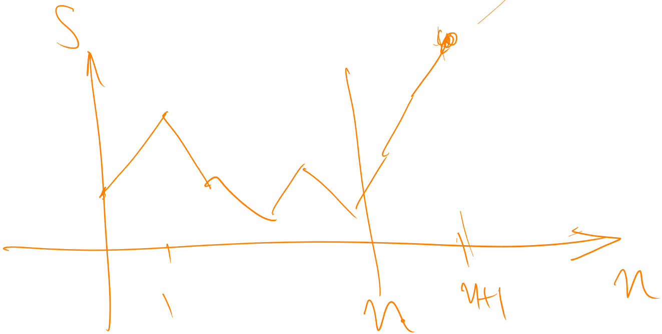
QED!!

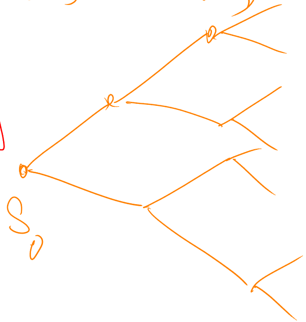
Question 7.15. *Is $Y_n = S_n$ a state process?* ← Yes (because S is markov)

Question 7.16. *Is $Y_n = \max_{k \leqslant n} S_n$ a state process?* ← NO

Question 7.17. *Is $Y_n = (S_n, \max_{k \leqslant n} S_n)$ a state process?* ← Yes!!

Y_n adapted? → Yes!





Let $M_n = \max_{0 \le k \le n} S_k$.

$Y_n = (S_n, M_n)$

$$M_{n+1} = \begin{cases} S_{n+1} & \text{if } S_{n+1} > M_n \\ M_n & S_{n+1} \le M_n \end{cases} =$$

can write in terms of S_n & ω_{n+1}

$\Rightarrow$ Can express Y_{n+1} ~~as~~ as a fn of Y_n & ω_{n+1}

$\Longrightarrow$ Y is a state process.

Question 7.18. *Let* $A_n = \sum_0^n S_k$. *Is* A_n *a state process?* NOT

Question 7.19. *Is* $Y_n = (S_n, A_n)$ *a state process?* ? ← Is

① M_g : $M_n = E_n M_{n+1} \iff \forall m \geq n,\ M_n = E_n M_m$

$$M_n = E_n M_{n+1} = E_n E_{n+1} M_{n+2} = E_n M_{n+2}$$