Last time:

RNP formula: $V_n = \frac{1}{P_n} \tilde{E}_n(D_N V_N)$ ⟵ not practical to compute with

(Computational time $\longrightarrow 2^N$)

Theorem 7.2. *Suppose a security pays* $V_N = \boxed{g(S_N)}$ *at maturity* N *for some (non-random) function* g*. Then the arbitrage free price at time* $n \leqslant N$ *is given by* $V_n = f_n(S_n)$*, where:*

(1) $f_N(x) = g(x)$ *for* $x \in \mathrm{Range}(S_N)$.

(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \boxed{\mathrm{Range}(S_n)}$.

Remark 7.3. Reduces the computational time from $O(2^N)$ to $O(\sum_0^N |\mathrm{Range}(S_n)|) = O(N^2)$ for the Binomial model.

Remark 7.4. Can solve this to get $f_n(x) = \sum_{k=0}^{N-n} \binom{N-n}{k} f_N(xu^k d^{N-n-k}) \cdot \frac{1}{(1+r)^{N-n}} \tilde{p}^k \tilde{q}^{N-n-k}$

① Compute $f_{N-1}(x) = \frac{1}{1+r}\left(\tilde{p} f_N(ux) + \tilde{q} f_N(dx)\right)$

② Compute $f_{N-2}(x) = \frac{1}{1+r}\left(\tilde{p} f_{N-1}(ux) + \tilde{q} f_{N-1}(dx)\right)$

$= \frac{1}{1+r}\Big(\tilde{p}\left(\frac{1}{1+r}\left(\tilde{p} f_N(u^2x) + \tilde{q} f_N(udx)\right)\right) +$

$+ \tilde{q}\left(\frac{1}{1+r}\left(\tilde{p} f_N(udx) + \tilde{q} f_N(d^2x)\right)\right)\Big)$

$$= \frac{1}{(1+r)^2}\left[\tilde{p}^2 f_N(u^2 x) + 2\tilde{p}\tilde{q} f_N(u d x) + \tilde{q}^2 f_N(d^2 x)\right]$$

③ Iterate this & get Rmk 7.4.

Question 7.5. *How do we handle other securities? E.g. Asian options (of the form $g(\sum_0^N S_k)$)?*

European call options $\rightarrow$ payoff $(S_N - K)^+$

Asian call option $\rightarrow$ payoff $\left(\frac{1}{N+1}\sum_0^N S_k - K\right)^+ \leftarrow$ Now

(American call $\rightarrow$ can exercise at any time $\leq N$) $\leftarrow$ IOU

Definition 7.6. We say a process X is a *Markov process* if $\boldsymbol{P}(X_{n+1} = x_{n+1} \mid X_1 = x_1, \dots, X_n = x_n) = \boldsymbol{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$.

Theorem 7.7. *A process X is Markov if and only if for every (bounded, continuous) function f, there exists a function g such that $\boldsymbol{E}_n f(X_{n+1}) = g(X_n)$.*

Question 7.8. *If X_n represents i.i.d. coin tosses, is X_n Markov? Is $Y_n = \sum_0^n X_k$ Markov?*

$\forall f$

$$E_n f(X_{n+1}) = E\big(f(X_{n+1}) \mid \mathcal{F}_n\big) \overset{\text{Markov}}{=} g(X_n)$$

$$E_n f(X_{n+1}) \overset{\text{indep}}{=} E f(X_{n+1}) \overset{\text{iid}}{=} E f(X_1) \quad (\text{some } \#)$$

$$(\text{Chose } g(x) = E f(X_1) \ \forall x)$$

$X_n \to$ iid. $\quad Y_n = \sum_0^n X_k \quad (Y_0 = 0)$

Q: Is Y markov

Guess $\to$ No

Guess $\to$ Yes!

Note: $Y_{n+1} = Y_n + X_{n+1}$

Compute $E_n f(Y_{n+1}) = E_n f(Y_n + X_{n+1})$

$$\underset{\text{indep lemma}}{=} \sum \underbrace{f(Y_n + x_i)}\ \underbrace{P(X_1 = x_i)}_{\text{#'s}}$$

$= \text{some fn of } Y_n. \Rightarrow \boxed{Y \text{ is Markov}}$

Question 7.9. *Is S_n (stock in the Binomial model) Markov under $\tilde{\boldsymbol{P}}$? Is $A_n = \frac{1}{n}\sum_0^n S_k$ Markov under $\tilde{\boldsymbol{P}}$?*

Guess S_n is markov.

Let $X_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$

$$\tilde{E}_n f(S_{n+1}) = \tilde{E}_n f(X_{n+1} S_n)$$

$$\overset{\text{indep lem}}{=} \tilde{p} f(u S_n) + \tilde{q} f(d S_n)$$

$$= \text{Same fn of } S_n. \implies S_n \text{ is Markov.}$$

Claim: A_n is NOT Markov

(Intuition $\rightarrow$ To find A_{n+1} need to know S_{n+1}

depends on S_n

Can't determine S_n in terms of A_n.)

Question 7.10. *Is* (S_n, A_n) *Markov?* (True → IOU)

Definition 7.11. We say a d-dimensional process $Y = (Y^1, \ldots, Y^d)$ process is a $\boxed{\textit{state process}}$ if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n. $(n \leq m)$

Remark 7.12. For state processes given $\boxed{f_N,}$ we find f_n by backward induction. The number of computations at time n is of order $\boxed{\text{Range}(Y_n).}$

Remark 7.13. The fact that $\boxed{S_n}$ is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process. (Binomial model $0<d<1+r<u$)

Pf of remk 7.13: Consider a sec with payoff $f_N(S_N)$

Let $n \leq N$. AFP at $n = V_n = \frac{1}{D_n} \tilde{E}_n D_N f_N(S_N)$

$n = N-1$: AFP $= V_{N-1} = \frac{1}{(1+r)} \tilde{E}_{N-1} f_N(S_N)$

$\underset{=}{\text{Markov}} \frac{1}{1+r}$ some fn (S_{N-1}) state at time $N-1$ (not history)