$\rightarrow S^1$ $u = 2$, $d = 1/2$

(Prob Heads $= 90\%$)

(Prob Tails $= 10\%$)

$C^1 \longrightarrow$ call option on S^1

$\Rightarrow S^2$ $u = 2$, $d = 1/2$

$P(\text{Heads}) = 99\%$

$P(\text{Tails}) = 1\%$

$C^2 \longrightarrow$ call option on S^2

$S^1_0 = S^2_0$ | Q: $C^1_0 = C^2_0$

7. State processes.

$\to u, d, r \quad 0 < d < 1+r < u.$

Question 7.1. *Consider the N-period binomial model, and a security with payoff $\boxed{V_N}$. Let X_n be the arbitrage free price at time $n \leqslant N$, and Δ_n be the number of shares in the replicating portfolio. What is an algorithm to find X_n, Δ_n for all $n \leqslant N$? How much is the computational time?*

① Know $\tilde{P}$: $\tilde{p} = \dfrac{1+r-d}{u-d}$ $\qquad \tilde{q} = \dfrac{u-(1+r)}{u-d}$.

$\hookrightarrow \tilde{P}(\omega_i = +1) = \tilde{p}$ & $\tilde{P}(\omega_i = -1) = \tilde{q}$ (RNM).

$X_N = V_N$ $\qquad$ (X_n = AFP = wealth of Rep Portfolio).

$D_n X_n$ is a $\tilde{P}$ mg. $\qquad$ ($D_n = (1+r)^{-n}$)

$$\Rightarrow \tilde{E}_n \left(D_{n+1} X_{n+1} \right) = D_n X_n$$

$$\Rightarrow \quad X_n = \frac{\tilde{E}_n X_{n+1}}{1+r}$$

$$X_n = X_n(\omega)$$ (Remember X_n only dep on $\omega_1, \dots, \omega_n$ & ω_{n+1} is indep of $\omega_1, \dots, \omega_n$)

$$X_n(\omega) = X_n(\omega_1, \omega_2, \dots, \omega_n) = \frac{\tilde{E}_n\left(X_{n+1}(\omega_1, \omega_2, \dots, \omega_n, \omega_{n+1})\right)}{1+r}$$

(Ind lemma)

$$= \frac{1}{1+r}\left[\tilde{p}\, X_{n+1}(\omega_1, \dots, \omega_n, 1) + \tilde{q}\, X_{n+1}(\omega_1, \omega_2, \dots, \omega_n, -1)\right]$$

$\Rightarrow$ To find X_N : Given $X_N = V_N$ (Given)

$$X_n(\omega_1, \omega_2, \cdots \omega_n) = \frac{1}{1+r}\left(\tilde{p}\, X_{n+1}(\omega_1, \cdots \omega_n, 1) + \tilde{q}\, X_{n+1}(\omega_1 \cdots \omega_n, -1)\right)$$

Backward Ind

For Δ_n : Last time

$$\Delta_n(\omega) = \frac{X_{n+1}(\omega_1, \cdots \omega_n, 1) - X_{n+1}(\omega_1 \cdots \omega_n, -1)}{(u-d)\, S_n(\omega)}$$

Computational Cost : $\approx O(2^N) \leftarrow$ Practical ?

(time)

Eg: $N = 100$. Cost $\approx 2^{100}$ operations

$$2^{10} \approx 10^{3} \Rightarrow 2^{100} \approx \underset{\sim}{10}^{30}$$

Guess: 10 bill op per sec. $\approx 10^{10}$ op per sec.

total time $\approx 10^{20}$ <u>seconds</u> (life of universe $\approx 10^{19}$ sec)

Goal $\longrightarrow$ Improve this. (Can do it in $\cancel{\boxed{O(N)}}$ or $O(N^2)$ time)

(Binom model)

Theorem 7.2. *Suppose a security pays $V_N = g(S_N)$ at maturity N for some $\boxed{\text{(non-random)}}$ function g. Then the arbitrage free price at time $n \leqslant N$ is given by $V_n = f_n(S_n)$, where:*

(1) $f_N(x) = V_N(x)$ *for* $x \in \mathrm{Range}(S_N)$. $g(x)$

(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \mathrm{Range}(S_n)$.

Remark 7.3. Reduces the computational time from $\boxed{O(2^N)}$ to $O(\sum_0^N \boxed{|\mathrm{Range}(S_n)|}) = \boxed{O(N^2)}$ for the Binomial model.

Remark 7.4. Can solve this to get $\underbrace{f_n(x) = \sum_{k=0}^{N-n} \binom{N-n}{k} f_N(xu^k d^{N-n-k})}$

Say $X_{n+1} = f_{n+1}(S_{n+1})$. (True for $n+1 = N$)

Know $X_n = X_n(\omega_1, \cdots \omega_n) = \frac{1}{1+r}\left(\tilde{p}\, X_{n+1}(\omega_1, \cdots \omega_n, +1) + \tilde{q}\, X_{n+1}(\omega_1, \cdots \omega_n, -1)\right)$

$= \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(S_{n+1}(\omega_1 \cdots \omega_n, +1)) + \tilde{q}\, f_{n+1}(S_{n+1}(\omega_1, \cdots \omega_n, -1))\right)$

$$\Rightarrow X_n(\omega) = \underbrace{\frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(S_n(\omega)\, u) + \tilde{q}\, f_{n+1}(S_n(\omega)\cdot d)\right)}_{\text{Some fn of } S_n\,!}$$

Let $x = S_n(\omega)$.

$$\boxed{\text{Let } f_n(x) = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(ux) + \tilde{q}\, f_{n+1}(dx)\right)}$$

Done!

Indep lemma: $X \to \mathcal{F}_n$ meas

$Y \to$ ind of $\mathcal{F}_n$.

$$E_n f(X, \underline{\underline{Y}}) = \sum_1^k f(X, \underline{\underline{y_i}}) \cdot \underbrace{P(Y = y_i)}$$