Recall : Binom model

$$S_{n+1} \begin{cases} \nearrow u\, S_n & \omega_{n+1} = 1 \\ \searrow d\, S_n & \omega_{n+1} = -1 \end{cases}$$

& $d < 1+r < u$ ($r \to$ int rate).

$\tilde{P} \to$ RNM.

$$\tilde{P}(\omega_i = 1) = \frac{1+r-d}{u-d} \in (0,1)$$

$$\tilde{P}(\omega_i = -1) = \frac{u-(1+r)}{u-d} \in (0,1).$$

$$\to \tilde{E}_n(D_{n+1} S_{n+1}) = D_n S_n$$

$\boxed{\text{Thm (IOU): } X_n \text{ is the wealth of a self fin Portfolio} \iff D_n X_n \text{ is a mg under } \tilde{P}.}$

Theorem 6.9. *Let $d < 1 + r < u$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays* $\boxed{V_N}$ *at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$\boxed{V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N).}$$

$\left(n = 0 : \quad V_0 = \tilde{E}(\cancel{D_N} V_N) \right)$

Pf: Let $X_n = \frac{1}{D_n} \tilde{E}_n(D_N V_N)$.

① $X_N = \frac{1}{D_N} \tilde{E}_N(D_N V_N) = V_N$.

② Note $D_n X_n = \tilde{E}_n(D_N V_N)$. Tower prop $\Rightarrow$ $\underbrace{D_n X_n \text{ is a mg under } \tilde{P}}$

$\Rightarrow$ (By thm) X_n = wealth of a self fin Portfolio at time n.

③ X_n = wealth of a Replicating Portfolio at time n.

$\Rightarrow X_n = \underbrace{AFP}_{V_n}$ $\Rightarrow$ QED.

Only used $D_n X_n$ a $\tilde{P}$ mg $\Rightarrow X$ self fin.

$\omega = (\omega_1, \omega_2, \dots \omega_N)$

Remark 6.10. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\underbrace{\Delta_n(\omega)}_{\text{\# shares of stock at time } n.} = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u-d)(S_n(\omega))} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u-d)(S_n(\omega))}$$

$\omega' = (\omega_1 \dots \omega_n)$

$\omega'' = (\omega_{n+2}, \dots \omega_N)$

Let $X_n = V_n -$

Holdings at time n: Δ_n shares of stock

$(X_n - \Delta_n S_n)$ cash.

Self fin $\Rightarrow$ $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

$X_{n+1}(\omega) \to$ only dep on 1^{st} $n+1$ coin tosses.

Write $X_{n+1}(\omega) = X_{n+1}(\omega', \omega_{n+1}, \cancel{\omega''}) = X_{n+1}(\omega', \omega_{n+1})$.

$$X_{n+1}(\omega', \omega_{n+1}) = \Delta_n(\omega') S_{n+1}(\omega', \omega_{n+1}) + (1+r)\big(X_n(\omega') - \Delta_n(\omega') S_n(\omega')\big)$$

2 Eq: $\omega_{n+1} = 1$ & $\omega_{n+1} = -1$

$$\begin{cases} X_{n+1}(\omega', 1) = \Delta_n(\omega') S_n(\omega')\, u + (1+r)\big(X_n(\omega') - \Delta_n(\omega') S_n(\omega')\big) \\ X_{n+1}(\omega', -1) = \text{''} \quad \text{''} \quad d + \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \end{cases}$$

subtract: $X_{n+1}(\omega', 1) - X_{n+1}(\omega', -1) = \Delta_n(\omega') S_n(\omega') (u - d)$

$(X_n = \underline{V_n})$

Proof of Theorem 6.5 part 1. Suppose X_n is the wealth of a self-financing portfolio. Need to show $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.

Assume $d < 1+r < u$. $\tilde{P} \to$ RNM. $\left(\tilde{E}_n (S_{n+1} D_{n+1}) = D_n S_n \right)$

$D_n = (1+r)^{-n}$

$X \to$ self fin.

$X_{n+1} = \Delta_n S_{n+1} + (X_n - \Delta_n S_n)(1+r)$

$\boxed{\text{NTS } D_n X_n \text{ is a } \tilde{P} \text{ mg}}$

$\Rightarrow D_{n+1} X_{n+1} = \Delta_n D_{n+1} S_{n+1} + (1+r) D_{n+1} (X_n - \Delta_n S_n)$

$\Rightarrow \tilde{E}_n (D_{n+1} X_{n+1}) = \Delta_n \tilde{E}_n (D_{n+1} S_{n+1}) + D_n (X_n - \Delta_n S_n)$

$= \Delta_n \cancel{D_n} S_n + D_n (X_n - \cancel{\Delta_n S_n}) = D_n X_n$ QED.

$$= P_n X_n$$

Proof of Theorem 6.5 part 2. $\boxed{\text{Suppose } D_n X_n \text{ is a martingale under } \tilde{\boldsymbol{P}}.}$ Need to show $\boxed{X_n \text{ is the wealth of a self-financing portfolio.}}$

$$X_{n+1}(\omega) = X_{n+1}(\omega', \omega_{n+1}, \omega'') \qquad \left(\begin{array}{l}\omega' = (\omega_1 \ldots \omega_n) \\ \omega'' = (\omega_{n+2}, \ldots \omega_N)\end{array}\right)$$

$$= X_{n+1}(\omega', \omega_{n+1})$$

$$\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} \in \mathbb{R}^2 \quad (\text{same vector}).$$

Write as a L.C. of $\begin{pmatrix} u \\ d \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Can always write $\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} = \underbrace{\underline{\Delta_n(\omega')} \, S_n(\omega') \begin{pmatrix} u \\ d \end{pmatrix}} + \Gamma_n(\omega') \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad X_{n+1}(\omega', \omega_{n+1}) = \Delta_n(\omega') \, S_{n+1}(\omega', \omega_{n+1}) + \Gamma_n(\omega').$$

$\Rightarrow$ Can always write $X_{n+1} = \Delta_n S_{n+1} + \underline{\Gamma_n}$ for some adapted processes Δ & Γ.

$$\boxed{\text{NTS}: \; \Gamma_n = (X_n - \Delta_n S_n)(1+r)}$$

Pf: Note $D_{n+1} X_{n+1} = \Delta_n D_{n+1} S_{n+1} + D_{n+1} \Gamma_n$

Know $D_n X_n$ is a $\tilde{P}$ mg.

$$\Rightarrow D_n X_n = \tilde{E}_n (D_{n+1} X_{n+1}) = \tilde{E}_n (RHS)$$

$$= \Delta_n (D_n S_n) + D_{n+1} \Gamma_n .$$

$$\Rightarrow \Gamma_n = \frac{1}{D_{n+1}} \left(D_n X_n - \Delta_n D_n S_n \right) = (1+r) X_n - \Delta_n S_n (1+r)$$

$$= (1+r)(X_n - \Delta_n S_n) \quad \text{QED.}$$