6. The multi-period binomial model

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $u, d > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1, \\ dS_n & \omega_{n+1} = -1. \end{cases}$
- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p_1 = \boldsymbol{P}(\omega_n = 1) > 0$ and $q_1 = 1 - p_1 = \boldsymbol{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $r > -1$.
- $D_n = (1+r)^{-n}$ be the discount factor (\$1 at time n is worth $\$D_n$ at time 0.)

Theorem 6.1. *There exists a (unique) equivalent measure* $\tilde{\boldsymbol{P}}$ *under which process* $\boxed{D_n S_n}$ *is a martingale* if and only if $\boxed{d < 1 + r < u}$. *In this case* $\tilde{\boldsymbol{P}}$ *is the probability measure obtained by tossing* N *i.i.d. coins with*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p}_1 = \frac{1 + r - d}{u - d}, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q}_1 = \frac{u - (1+r)}{u - d}.$$

Definition 6.2. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure*.

Remark 6.3. If there are more than one risky assets, $S^1, \ldots, S^k$, then we require $D_n S_n^1, \ldots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.* (Δ_n must be $\mathcal{F}_n$ meas)
- Continuing further, we see $\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n).}$
- *Both X and Δ are adapted processes.*

Definition 6.4. A *self-financing portfolio* is a portfolio whose wealth evolves according to (X_n is adapted)

$$\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,}$$

for some adapted process Δ_n.

Theorem 6.5. *Let $d < 1 + r < u$, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.6. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

Before proving Theorem 6.5, we consider a few consequences:

Theorem 6.7. *The multi-period binomial model is arbitrage free if and only if* $\boxed{d < 1 + r < u}$.

Remark 6.8. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

$\Rightarrow$ Pf: (1) Market has no arbitrage if $X_0 = 0$

$X_n \to$ wealth at time n of a self-financing portfolio.

$\underbrace{X_0 = 0, \; X_n \geq 0}_{\text{no risk (almost surely)}} \Rightarrow X_N = 0$ (almost surely)

(2) Pf: If $d \geq 1+r$ or $u \leq 1+r \Rightarrow$ market has arb. (Lec 2)

(3) Say X_n is the wealth a self financing portfolio.

Suppose $X_N \geq 0$ almost surely & $X_0 = 0$

NTS $X_N = 0$.

Knows $D_n X_n$ is a mg under $\tilde{P}$.

(∵ mg's have const exp) $\Rightarrow$ $\tilde{E} D_N X_N = \tilde{E} D_0 X_0 = 0$

$\Rightarrow$ $\tilde{E}(1+r)^{-N} X_N = 0$

Note $(1+r)^{-N} X_N \geq 0$

$\Rightarrow (1+r)^{-N} X_N = 0$ almost surely

$\Rightarrow X_N = 0$ almost surely QED.

Only used: If X_n is self-fin then $D_n X_n$ is a mg under $\tilde{P}$

(Risk Neutral pricing formula)

Theorem 6.9. *Let $d < 1 + r < u$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays V_N at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_N V_N)\,. = \tilde{E}_n\left(\frac{D_N V_N}{D_n}\right)$$

($\because D_n$ is $\mathcal{F}_n$ - meas)

Pf: $D_n V_n = \boxed{\tilde{E}_n(D_N V_N)}$

Claim: Let $M_n = \tilde{E}_n(D_N V_N)$. Q: Is M a mg under $\tilde{P}$?

Claim: M is a mg under $\tilde{P}$. (Pf: $\tilde{E}_n M_{n+1} = \tilde{E}_n(\tilde{E}_{n+1} D_N V_N) \overset{\text{tower}}{=} \tilde{E}_n(D_N V_N) = M_n$. QED)

Let $V_n = \frac{1}{D_n}\tilde{E}_n(D_N V_N) \Rightarrow \underbrace{D_n V_n}_{} = \underbrace{\tilde{E}_n(D_N V_N)}_{} = M_n$ (mg under $\tilde{P}$)

By Thm 6.5: $\underline{\underline{V_n}}$ is the wealth of a self financing portfolio.

$V_N \longrightarrow$ Payoff of any security $\Rightarrow$ V_n = wealth at time n of a R. Portfolio

$\Rightarrow$ (Wealth of R. Portfolio) V_n = AFP of the security.

QED.

(Note: in this thm we used if $D_n X_n$ is a mg under $\tilde{P}$, then we must have X_n = wealth of a self financing portfolio).

IOU: Pf of Thm 6.5.

Remark 6.10. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{u-d} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{u-d}$$

Prop: If M is a mg then $EM_n = EM_0 \ \forall n$.

↳ Pf: Claim $EM_n = EM_{n+1}$ (& use induction)

Know $M_n = E_n M_{n+1} \Rightarrow EM_n = \underbrace{E E_n}_{E} M_{n+1}$

$E E_n X = E X$

EM_{n+1} QED.