Last time: Change of measure:

Ω $\quad p$ (PMF) $\quad P(A) = \sum_{\omega \in A} p(\omega)$

$\hookrightarrow$ (input) $\tilde{p}$ (new PMF) $\quad \tilde{P}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$

① P & $\tilde{P}$ are equiv: $P(A) = 0 \iff \tilde{P}(A) = 0$

if (alternately $p(\omega) = 0 \iff \tilde{p}(\omega) = 0$)

$X_{n+1} = X_n + \underbrace{\omega_{n+1}}_{\pm 1} + a$ $\quad (a \in \mathbb{R})$. $\to$ $a \neq 0 \Rightarrow X$ is not a mg under P

Find $\tilde{P}$ under which X is a mg!

Example 5.50. Suppose now $\boldsymbol{P}(\omega_n = 1) = p$ and $\boldsymbol{P}(\omega_n = -1) = q = 1 - p$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $D_n = (1+r)^{-n}$ be the "discount factor". Find an equivalent measure under which $D_n S_n$ is a martingale.

$$(\text{Value of cash at time } n) \cdot \underbrace{D_n}_{(1+r)^{-n}} = \text{value of cash at time } 0$$

Q: Find $\tilde{P}$ ∋ $\tilde{P}$ is equiv to P & $\underline{D_n} S_n$ is a mg.

Say mdl $\tilde{P}$ $\to$ $\tilde{P}(\omega_i = 1) = \tilde{p}_1$

& $\tilde{P}(\omega_i = -1) = 1 - \tilde{p}_1 = \tilde{q}_1$

Compute $\tilde{E}_n(D_{n+1} S_{n+1})$ (Want $= D_n S_n$)

$$X_n = X(\omega) = \begin{cases} u & \text{if } \omega_n = 1 \\ d & \text{if } \omega_n = -1 \end{cases} \qquad (\omega = (\omega_1, \omega_2 \cdots \omega_N))$$

Then $\quad S_{n+1} = X_{n+1} S_n$

$$\Rightarrow \tilde{E}_n(D_{n+1} S_{n+1}) = (1+r)^{-(n+1)} \tilde{E}_n(X_{n+1} S_n)$$

$$= D_{n+1} S_n \tilde{E}_n X_{n+1} \qquad (\because S_n \text{ is } \mathcal{F}_n \text{ meas})$$

$$= D_{n+1} S_n \tilde{E} X_{n+1} \qquad (\because X_{n+1} \text{ is ind of } \mathcal{F}_n \text{ under } \tilde{P})$$

$M_n = D_n S_n$

$$= D_{n+1} S_n \left(u \tilde{p}_1 + d(1-\tilde{p}_1) \right) \overset{\text{Want}}{=} D_n S_n$$

Choose $\tilde{p}_1$ so that $(1+r)^{-(n+1)} \cancel{S_n} \left(u \tilde{p}_1 + d \tilde{q}_1 \right) = (1+r)^{-n} \cancel{S_n}$

$$\boxed{\Rightarrow \quad u \tilde{p}_1 + d \tilde{q}_1 = 1+r}$$

Solve: $\tilde{p}_1 (u-d) + d = 1+r \iff \boxed{\tilde{p}_1 = \dfrac{(1+r) - d}{u-d}}$

Will only give an equiv meas $\iff d < 1+r < u$

6. The multi-period binomial model

Example 6.1 (Binomial model revisited). Assume $\Omega = \{\pm 1\}^N$. Let $u, d > 0$, $S_0 > 0$. Define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1, \\ dS_n & \omega_{n+1} = -1. \end{cases}$

- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p\boldsymbol{P}(\omega_n = 1) > 0$ and $q = 1 - p = \boldsymbol{P}(\omega_n = -1) > 0$.
- Let $r > -1$ be the interest rate, and $D_n = (1+r)^{-n}$ be the discount factor.

Theorem 6.2. *There exists a (unique) equivalent measure $\tilde{\boldsymbol{P}}$ under which process $D_n S_n$ is a martingale* if and only if $d < 1 + r < u$. *In this case $\tilde{\boldsymbol{P}}$ is given by:*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p} = \frac{1 + r - d}{u - d}, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q} = \frac{u - (1+r)}{u - d}.$$

Definition 6.3. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.4. If there are more than one risky assets, $S^1, \dots, S^k$, then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{P}$.

Find $\tilde{P}$ done.

$D_n S$

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.*
- Continuing further, we see $X_{n+1} = \Delta_n \boxed{S_{n+1}} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

Theorem 6.5. *The discounted wealth $\boxed{D_n X_n}$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.6. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.