2. $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. X independent of $\mathcal{F}_n$, $Y \in \mathcal{F}_n$.

$Z = XY$, $\mathbb{E}_n[Z] = 0$. $\quad \mathbb{E}_n[XY] = \mathbb{E}_n[X]\,Y$

$\xrightarrow{\text{Thm. 5.28}} \mathbb{E}_n[X] = \mathbb{E}[X]$. $\quad \Rightarrow \mathbb{E}[X]\,Y = 0$.

4. (a) $\mathbb{E}[M_n^2] = \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \boxed{\mathbb{E}[X_i X_j]}$

① $i \neq j$, X_i independent of X_j, $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\,\mathbb{E}[X_j] = 0$

② $i = j$. $\mathbb{E}[X_i^2] = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1$

$\mathbb{E}[M_n^2] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] = n$. $\leftarrow$

$\mathbb{E}[X_i^2] = p(H)\cdot(X_i(H))^2 + p(T)\cdot(X_i(T))^2$

(b) $c, \sigma > 0$ s.t. $\mathbb{E}_n[S_{n+1}] = S_n$

$$\mathbb{E}_n\left[c^{n+1} e^{\sigma M_{n+1}}\right] = c^{n+1}\,\mathbb{E}_n\left[e^{\sigma(M_n + X_{n+1})}\right] = c^{n+1}\,\mathbb{E}_n\left[\overbrace{e^{\sigma M_n}}^{\in \mathcal{F}_n} \cdot e^{\sigma X_{n+1}}\right]$$

$$= c^{n+1} e^{\sigma M_n}\,\mathbb{E}_n\left[e^{\sigma X_{n+1}}\right] = \underbrace{c^{n+1} \cancel{e^{\sigma M_n}} \cdot \left(\frac{1}{2} e^{\sigma} + \frac{1}{2} e^{-\sigma}\right)} = c^{n} \cancel{e^{\sigma M_n}}$$

$$c\left(\frac{1}{2} e^{\sigma} + \frac{1}{2} e^{-\sigma}\right) = 1$$

$$\mathbb{E}_n\left(e^{\sigma X_{n+1}}\right) = \mathbb{E}\left(e^{\sigma X_{n+1}}\right) = p(H)\cdot e^{\sigma(\overset{1}{X_{n+1}(H)})} + p(T)\cdot e^{\sigma \overset{-1}{X_{n+1}(T)}}$$

$$= \frac{1}{2}\left(e^{\sigma} + e^{-\sigma}\right)$$

3.

$$\mathbb{E}_n(f(X,Y)) = \sum_{i=1}^{m} f(x_i, Y)\,\mathbb{P}[X = x_i] \qquad \{x_1, \dots, x_m\} = X(\Omega).$$

① $\sum_{i=1}^{m} f(x_i, Y)\,\mathbb{P}[X = x_i] \in \mathcal{F}_n$

② $\forall A \in \mathcal{F}_n, \quad \sum_{\omega \in A} \mathbb{E}_n[f(X,Y)(\omega)]\,p(\omega) = \sum_{\omega \in A} f(X,Y)(\omega)\,p(\omega)$

① . ~~fix~~ fix x_i, $f(x_i, Y) \in \mathcal{F}_n$, $\mathbb{P}[X = x_i]$ constant.

② $\sum_{\omega \in A} \left(\sum_{i=1}^{m} f(x_i, Y)\,\mathbb{P}[X = x_i] \right)(\omega)\, p(\omega) = \sum_{\omega \in A} f(X,Y)(\omega)\,p(\omega)$

$$\sum_{\omega \in A} \sum_{i=1}^{m} f(x_i, Y(\omega))\,\mathbb{P}[X = x_i]\; P(\omega)$$

$$\mathbb{E}[g(X)] = \sum_{i=1}^{m} \mathbb{P}[X = x_i]\, g(x_i)$$

$$= \sum_{i=1}^{m} \mathbb{P}[X = x_i] \left(\sum_{\omega \in A} f(x_i, Y(\omega))\,P(\omega) \right)$$

$Z = g(X) = g(x_i)$. ~~$Z(x_i)$~~

$$= \mathbb{E}\left[\sum_{\omega \in A} f(X, Y(\omega))\,P(\omega) \right] = \mathbb{E}_n\left[\sum_{\omega \in A} f(X, Y(\omega))\,P(\omega) \right]$$

$\in \mathcal{F}_n$

$$= \sum_{\omega \in A} \mathbb{E}_n[f(X, Y(\omega))\,P(\omega)] = \sum_{\omega \in A} \mathbb{E}_n(f(X, Y(\omega)))\,P(\omega)$$

given $\omega \in A \subseteq \mathcal{F}_n$, $Y(\omega)$, $P(\omega)$ constant.
fix

$$= \sum_{\omega \in A} f(X, Y(\omega))\,P(\omega).$$

$$= \sum_{\omega \in A} f(X,Y)(\omega) \cdot P(\omega).$$

$X(\omega) = X$, $\omega \in \mathcal{F}_n$.

Def: ① $\mathbb{E}_n X \in \mathcal{F}_n$

② $\sum_{\omega \in A} \mathbb{E}_n X(\omega)\, p(\omega) = \sum_{\omega \in A} X(\omega)\, p(\omega)$.

2. example: toss fair coin twice. $\mathcal{F}_n = \mathcal{F}_1$.

$$X = \begin{cases} 1 & \text{Heads on second} \\ -1 & \text{T on second} \end{cases}$$

$$Y = \begin{cases} 1 & \text{H on first} \\ -1 & \text{T on first} \end{cases}$$

$XY = Z$ ..

4. (a) $M_n = \sum_{j=1}^{n} X_j$. $\quad \mathbb{E}[X_n] = \frac{1}{2}\cdot 1 + \frac{1}{2}(-1) = 0.\ \forall n.$

$$\mathbb{E}_n[M_{n+1}] = \mathbb{E}_n\Big[\sum_{j=1}^{n+1} X_j\Big] = \mathbb{E}_n\Big[\Big(\underbrace{\sum_{j=1}^{n} X_j}_{\in \mathcal{F}_n}\Big) + X_{n+1}\Big].$$

$$= \sum_{j=1}^{n} X_j + \mathbb{E}_n[X_{n+1}] = M_n + \mathbb{E}[X_{n+1}] = M_n + 0 = M_n.\ ,\ \underline{\text{MRT}}$$

$$\mathbb{E}[M_n] = \mathbb{E}\big[\mathbb{E}_{n-1}[M_n]\big] = \mathbb{E}[M_{n-1}],\ \forall n.$$

$$\mathbb{E}[M_n] = \mathbb{E}[M_{n-1}] = \cdots = \mathbb{E}[M_1] = \mathbb{E}[X_1] = 0.$$

$$\mathbb{E}[M_n] = \mathbb{E}\Big[\sum_{j=1}^{n} X_j\Big] = \sum_{j=1}^{n} \mathbb{E}[X_j] = 0.$$