Q1: $X_{n+1} = \begin{cases} u & \text{if } \omega_{n+1} = 1 \\ d & \text{if } \omega_{n+1} = -1 \end{cases}$

$S_{n+1} = X_{n+1} S_n \Rightarrow E_n S_{n+1} = S_n \underbrace{E X_{n+1}}_{pu + qd}$

$E_n S_{n+2} \overset{\text{tower}}{=} E_n E_{n+1} S_{n+2}$

$E_n \left[S_{n+1} (pu + qd) \right]$

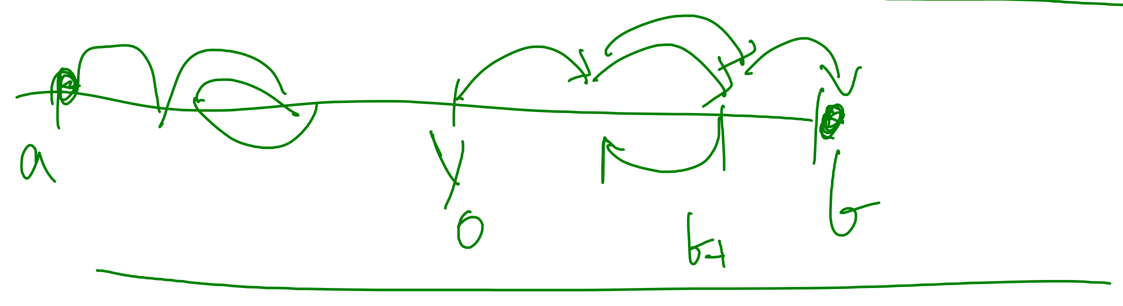
$= (pu + qd)^2 S_n$

$E_n S_N = (\quad'' \quad)^{N-n} S_n$ (induct)

(b) $1 + r = pu + qd.$

Q5: $Y_{n+1} = Y_n + X_{n+1}$ $\quad X_n = \begin{cases} 1 & \omega_n = \text{Heads} \\ -1 & \omega_n = \text{tails} \end{cases}$

$\to Y_{n+1} = Y_n$ if $Y_n \in \{a, b\}$ | If $Y_n \in (a, b)$



(a) Find $E_n Y_{n+1}$

① $Y_{n+1} = Y_n + \mathbb{1}_{\{Y_n \in (a,b)\}} X_{n+1}$

Say $A \subseteq \Omega$. Define $\mathbb{1}_A$ by $\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

② $E_n Y_{n+1} = E_n \left(Y_n + \mathbb{1}_{\{Y_n \in (a,b)\}} X_{n+1} \right)$

$= E_n Y_n + E_n \left(\mathbb{1}_{\{Y_n \in (a,b)\}} X_{n+1} \right)$

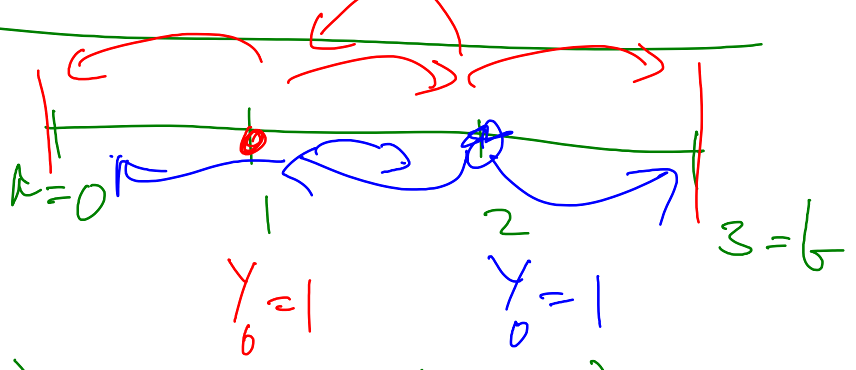
($\because \mathbb{1}_{Y_n \in (a,b)}$ is $\mathcal{F}_n$-meas)

$= Y_n + \mathbb{1}_{\{Y_n \in (a,b)\}} E_n X_{n+1}$

$= Y_n + \mathbb{1}_{\{Y_n \in (a,b)\}} \underbrace{E_n X_{n+1}}_{\underbrace{E X_{n+1}}_{0}}$

$\boxed{\Rightarrow E_n Y_{n+1} = Y_n}$

5b:



① $Y_0 = 1$

$p_n = P(Y_n = 0)$ (given $Y_0 = 1$)

② $q_n = P(Y_n = 0)$ (given $Y_0 = 2$)

$p_{n+2} = P(Y_{n+2} = 0, \text{ given } Y_0 = 1)$

$p_{n+2} =$ $\quad Y_0 = 1$

At time 1 $\begin{cases} Y_1 = 0 & (\text{prob } \tfrac{1}{2}) \\ Y_1 = 2 & (\text{prob } \tfrac{1}{2}) \end{cases}$

If $Y_1 = 0$, then $P(Y_{n+2} = 0) = 1$

If $Y_1 = 2$ then $P(Y_{n+2} = 0) = q_{n+1}$

$\boxed{\Rightarrow p_{n+2} = \frac{1}{2} + \frac{q_{n+1}}{2}}$

Find q_{n+1} in terms of p_n / q_n ...

$q_{n+1} = P(Y_{n+1} = 0, \text{ given } Y_0 = 2)$

at time 1: $Y_1 = 1$ prob $1/2$

$Y_1 = 3$ $\quad$ " $\quad 1/2$.

at time $n+1$: $Y_{n+1} = 0$ with prob p_n if $Y_1 = 1$

$\to Y_{n+1} = 0$ with prob 0 if $Y_1 = 3$

$\boxed{\Rightarrow q_{n+1} = \frac{1}{2} p_n + \frac{1}{2} \cdot 0}$

$\Rightarrow p_{n+2} = \underline{\qquad\qquad}$ (in terms of p_n)