Last time: Martingales $\longrightarrow$ M is a mg if $\underbrace{E_n M_{n+1}} = M_n$

(fair game)

5.5. Change of measure.

- Let $p\colon \Omega \to [0,1]$ be a probability mass function on Ω, and $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$ be the probability measure.
- Let $\tilde{p}\colon \Omega \to [0,1]$ be another probability mass function, and define a second probability measure $\tilde{\boldsymbol{P}}$ by $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$.

Definition 5.47. We say $\boldsymbol{P}$ and $\tilde{P}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Remark 5.48. When Ω is finite, $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if and only if we have $p(\omega) = 0 \iff \tilde{p}(\omega) = 0$ for all $\omega \in \Omega$.

We let $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ denote the expectation and conditional expectations with respect to $\tilde{\boldsymbol{P}}$ respectively.

$\Omega \xrightarrow{p} [0,1]$ $\quad (\sum_{\omega \in \Omega} p(\omega) = 1)$

$A \subseteq \Omega \quad P(A) = \sum_{\omega \in A} p(\omega)$

$\Omega \xrightarrow{\tilde{p}} [0,1]$ $\quad (\sum_{\omega \in \Omega} \tilde{p}(\omega) = 1)$

$A \subseteq \Omega, \ \tilde{P}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$

① X a RV. $\quad EX = \sum_{\omega \in \Omega} X(\omega)\, p(\omega)$ (Expected value under P)

② Given the new PMF $\tilde{p}$, define $\tilde{E}X = \sum_{\omega \in \Omega} X(\omega)\, \tilde{p}(\omega)$ (Expected val under $\tilde{P}$)

③ $E_n X = E(X \mid \mathcal{F}_n) =$ cond exp of X given $\mathcal{F}_n$

ⓐ $E_n X$ is an $\underline{\mathcal{F}_n}$ meas RV.

ⓑ $\forall A \in \underline{\mathcal{F}_n}$ $\quad \sum_{\omega \in A} E_n X(\omega)\, p(\omega) = \sum_{\omega \in A} \underline{X(\omega)}\, p(\omega)$

④ $\tilde{E}_n X \longrightarrow \tilde{E}(X \mid \mathcal{F}_n) \longrightarrow$ cond exp of X given $\mathcal{F}_n$

ⓐ $\tilde{E}_n X$ is an $\mathcal{F}_n$-meas RV

ⓑ $\forall A \in \mathcal{F}_n$, $\quad \sum_{\omega \in A} \tilde{E}_n X(\omega)\, \tilde{p}(\omega) = \sum_{\omega \in A} X(\omega)\, \tilde{p}(\omega)$

Example 5.49. Let Ω be the sample space corresponding to N i.i.d. fair coins (heads is 1, tails is -1). Let $a \in \mathbb{R}$ and define $X_{n+1}(\omega) = X_n(\omega) + \omega_{n+1} + a$. For what a is there an equivalent measure $\tilde{P}$ such that X is a martingale?

$X_0 = 0$, $\quad X_1 = X_0 + \omega_1 + a$ $\qquad$ ($a \in \mathbb{R}$ same const).

$X_2 = X_1 + \omega_2 + a$

Q: Is X a mg? (Fair coin X is a mg $\iff$ $a = 0$

$$E_n X_{n+1} = E_n(X_n + \omega_{n+1} + a) = X_n + \underbrace{E_n \omega_{n+1}}_{} + \overbrace{E_n a}^{a}$$

$$E_n X_{n+1} = X_n + 0 + \underline{\underline{a}}$$

$E\,\omega_{n+1} = 0$

(ω_{n+1} is indep of $\mathcal{F}_n$)

Goal: Find $\tilde{P}$ so that P & $\tilde{P}$ are equiv & X is a mg under $\tilde{P}$.

Let $\tilde{p}$ be a PMF (will find $\tilde{p}$ shortly)

$$\tilde{E}_n X_{n+1} = \tilde{E}_n(X_n + \omega_{n+1} + a) = X_n + \underbrace{\tilde{E}_n \omega_{n+1} + a}$$

Need to chose $\tilde{p}$ so that $\boxed{\tilde{E}_n \omega_{n+1} + a = 0}$

Say we choose $\tilde{p}$ so that the coins are iid &

$$\tilde{P}(\omega_n = 1) = \tilde{p}_1 \quad \& \quad \tilde{P}(\omega_n = -1) = \tilde{q}_1 = 1 - \tilde{p}_1.$$

Red Box $\rightarrow$ $\tilde{E}_n \omega_{n+1} + a = 0 \iff \tilde{E}\omega_{n+1} + a = \tilde{p}_1 1 + (1-\tilde{p}_1)(-1) + a = 0$

$\iff 2\tilde{p}_1 - 1 + a = 0$

$$\boxed{\iff \tilde{p}_1 = \frac{1-a}{2}}$$

① need $\tilde{p}$ to be a PMF

② need $\tilde{P}$ equiv to P

$(p(\omega) = 0 \iff \tilde{p}(\omega) = 0)$

$\tilde{p}(\omega) = \tilde{p}(\omega_1, \omega_2 \dots \omega_N) = (\tilde{p}_1)^{\# \text{heads}} (1-\tilde{p}_1)^{\# \text{tails}}$

Note $\tilde{p}_1 \in (0,1) \iff a \in (-1,1)$

Let $\tilde{P}$ be the Pr meas where each coin comes up heads with prob. $\tilde{p}_1 = \frac{1-a}{2} \in (0,1)$

Then X is a mg under $\tilde{P}$!

Note $p_1 \in (0,1) \iff \tilde{P}$ & P are eq

Example 5.50. Suppose now $\boldsymbol{P}(\omega_n = 1) = p$ and $\boldsymbol{P}(\omega_n = -1) = q = 1 - p$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $D_n = (1 + r)^{-n}$ be the "discount factor". Find an equivalent measure under which $D_n S_n$ is a martingale.