## 5.4. Martingales.

← think of $X_n$ = RV at time $n$

**Definition 5.30.** A *stochastic process* is a collection of random variables $X_0, X_1, \ldots, X_N$.

**Definition 5.31.** A stochastic process is *adapted* if $X_n$ is $\mathcal{F}_n$-measurable for all $n$. (Non-anticipating.)

**Question 5.32.** *Is $X_n(\omega) = \sum_{i \leqslant n} \omega_i$ adapted?* YES ($\omega_i \in \pm 1$ ($i^{th}$ coin toss))

**Question 5.33.** *Is $X_n(\omega) = \omega_n$ adapted? Is $\boxed{X_n(\omega) = 15}$ adapted? Is $X_n(\omega) = \omega_{15}$ adapted? Is $X_n(\omega) = \omega_{N-i}$ adapted?*

*Remark* 5.34. We will always model the price of assets by *adapted* processes. We will also only consider trading strategies which are adapted.

$\omega = (\omega_1, \omega_2, \ldots \omega_N)$

YES.

YES

& $X_0$ is $\mathcal{F}_0$ → meas, $\mathcal{F}_0 = \{\phi, \Omega\}$

① A RV is $\mathcal{F}_0$ meas $\iff$ it is const.

② Constants are $\mathcal{F}_n$-meas $\forall n \geq 0$.

③ $X_n$ adapted $\Rightarrow$ $X_n$ is $\mathcal{F}_n$ meas $\forall n$. Since $\mathcal{F}_n \subseteq \mathcal{F}_m$ $\forall m \geq n \Rightarrow$

$X_n$ is $\mathcal{F}_m$-meas $\forall\, m \underline{\underline{\geq}}\, n$

Q: $X_n = \omega_{15}$ $\forall\, n$. Is $X$ adapted? NO

*Example* 5.35 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here $r$ is the interest rate.) (Adapted)

*Example* 5.36. Suppose $\Omega = \{\pm 1\}^N \cong \{H, T\}^N \cong \{1, 2\}^N$. Let $S_0 = a \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$
Is $S_n$ adapted? (Used to model stock price in the multi-period Binomial model.)

$\omega = (\omega_1, \omega_2, \dots, \omega_N)$

Yes.

**Definition 5.37.** We say an adapted process $X_n$ is a martingale if $\boldsymbol{E}_n X_{n+1} = X_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

*Remark* 5.38. Intuition: A martingale is a "fair game".

**Question 5.39.** *If $m \leqslant n$, is $\boldsymbol{E}_m X_n = X_m$?* ← YES

$$E_n X_{n+1} = X_n$$

Best approx of $X_{n+1}$ given only the first $n$ - die rolls is $X_n$ itself!

Q: $m \le n$: $E_m X_n = X_m$?

$$E_n X_{n+2} \underset{\text{Tower}}{=} E_n \underbrace{E_{n+1} X_{n+2}}_{\text{mg}} \overset{\text{mg}}{=} E_n ( X_{n+1} ) \overset{\text{mg}}{=} X_n .$$

QED

**Question 5.40.** *If $M$ is a martingale does $\boldsymbol{E}M_n$ change with $n$?*

Know $E_n M_{n+1} = M_n$ $\quad (\Leftrightarrow \forall m \le n,\ E_m M_n = M_m)$

Claim: $E M_{n+1} = E M_n$ $\quad$ ($E$ not $E_n$).

Pf: $E M_{n+1} \underset{\substack{\uparrow \\ \text{def of } E_n}}{=} E E_n M_{n+1} \underset{\substack{\uparrow \\ \text{mg}}}{=} E M_n$ QED.

$\forall A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$ $\qquad E(E_n X) = EX$

$\Omega \in \mathcal{F}_n$. Put $A = \Omega$. $\Rightarrow \displaystyle\sum_{\omega \in \Omega} E_n X(\omega) p(\omega) = \sum_{\omega \in \Omega} X(\omega) p(\omega)$

**Question 5.41.** *Conversely, if $\boldsymbol{E}M_n$ is constant, is $M$ a martingale?* NO!! (IOU Eg there are many).

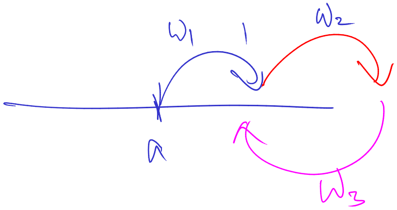
Eg 0: $M_n = 0 \ \forall n$ is a mg.

*Example* 5.42. Unbiased random walks are martingales.

Say $\omega_i \longrightarrow$ outcome of a fair coin $\begin{cases} \omega_i = 1 & \text{w/ prob } 1/2 \\ \omega_i = -1 & \text{'' } 1/2. \end{cases}$

$\omega = (\omega_1, \dots, \omega_N) \longleftarrow N$ iid fair coins.

$X_0 = a \in \mathbb{R}$. $\quad X_{n+1}(\omega) = X_n(\omega) + \omega_{n+1} = a + \sum_{k=1}^{n+1} \omega_k$




Claim: $X_n$ is a mg.

Pf: (0) $X$ adapted (Yes)

lin.

(1) $E_n X_{n+1} = E_n(X_n + \omega_{n+1}) = \underbrace{E_n X_n}_{X_n} + E_n \omega_{n+1}$

$$= X_n + E W_{n+1}$$ ($\because W_{n+1}$ ind of $\mathcal{F}_n$)

($\because X_n$ is $\mathcal{F}_n$ - meas)

$$= X_n + 0$$

QED

*Example* 5.43. More generally, if $M_{n+1} - M_n$ is mean 0 and independent of $\mathcal{F}_n$, then $M$ is a martingale. ← (indep increments)

**Question 5.44.** *If $M$ is a martingale, must $M_{n+1} - M_n$ be independent of $\mathcal{F}_n$?* NO! $M$ mg $\not\Rightarrow$ $M_{n+1} - M_n$ is indep of $\mathcal{F}_n$. HW3 Q2

Pf: ① Assume $M_{n+1} - M_n$ is ind of $\mathcal{F}_n$

② $E(M_{n+1} - M_n) = 0$

$\Rightarrow M$ is a mg.

Pf: $E_n M_{n+1} = E_n(M_{n+1} - M_n + M_n) = \underbrace{E_n(M_{n+1} - M_n)}_{= E(M_{n+1} - M_n) = 0 \text{ (: ind)}} + \underbrace{E_n M_n}_{= M_n \text{ (adapted)}}$

QED

Markov: $E_n \cancel{f}(X_{n+1}) = \cancel{g}(X_n)$