Last time: Cond Exp prop: $\boxed{E_n X = E(X \mid \mathcal{F}_n)}$

(Def) ① $E_n X \to$ ⓐ $\mathcal{F}_n$ - meas RV & ⓑ $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$

→ ② $E_n(X + \alpha Y) = E_n(X) + \alpha E_n(Y)$ (X, Y RV's & $\alpha \in \mathbb{R}$) (on HW)

→ ③ (Tower) $m \le n \Rightarrow E_m(E_n X) = E_m(X)$

④ If X is $\mathcal{F}_n$ meas & Y is anything then $E_n(XY) = X(E_n Y)$

Theorem 5.28. *If X is $\boxed{\text{independent}}$ of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$* (almost surely)

(If X is $\mathcal{F}_n$-meas, then $E_n X = X$)

X is indep of the first n die rolls

① Def: We say X is ind of $\mathcal{F}_n$ if $\forall A \in \mathcal{F}_n$ & $B \subseteq \mathbb{R}$ the events A & $\{X \in B\}$ are ind.

$\underbrace{A \in \mathcal{F}_n}_{\text{events det from } 1^{st}\ n \text{ die rolls}}$

$B \subseteq \mathbb{R}$

$\{X \in B\} \leftarrow$ some event

$\underbrace{\{\omega \mid X(\omega) \in B\}}_{\text{evts det by } X}$

Remk: Ω finite $\Rightarrow$ ① $\iff$ $\forall x \in \mathbb{R}$ $\forall A \in \mathcal{F}_n$ the events A & $\{X = x\}$ are ind

Thm: X ind of $\mathcal{F}_n \Rightarrow \underbrace{E_n X}_{\mathcal{F}_n\text{-meas RV}} = \underbrace{EX}_{\text{number}}$ (almost surely)

Conv is false! $E_n X = EX \not\Rightarrow X$ is ind of $\mathcal{F}_n$

i.e. $E_n X$ is the const EX almost surely!

Pf: ① Check EX is an $\mathcal{F}_n$-meas RV (true! indep of all div rolls).

② Check $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} \underbrace{\cancel{E_n} X(\omega)}_{EX \text{ (no } n)} p(\omega) = \underbrace{\sum_{\omega \in A} X(\omega) p(\omega)}$

Pf: Say X takes on values $x_1 \ldots x_m$

Then $\displaystyle\sum_{\omega\in A} X(\omega)p(\omega) = \sum_{k=1}^{m} x_k \underbrace{P(\{X=x_k\}\cap A)}$

$$= \sum_{k=1}^{m} x_k P(X=x_k)\,P(A) \qquad (\because X \text{ is ind of } \mathcal{E}_n)$$

$$= P(A) \underbrace{\sum_{k=1}^{m} x_k P(X=x_k)}_{EX} = P(A)(EX)$$

$$= \sum_{\omega\in A} \underline{\underline{p(\omega)\,EX}} \qquad \text{QED !!}$$

Indep lemma:

1. X is $\mathcal{F}_n$ meas $\Rightarrow \overline{E_n X = X}$ (a.s)
2. Y indep of $\mathcal{F}_n$ $\Rightarrow E_n Y = EY$ (a.s.)
3. Q: $E_n f(X, Y)$ for some fn f (e.g. $E_n (X+Y)^{10}$)

"leave alone" "average Y"

Claim: $E_n f(X,Y) = \overbrace{\sum_k f(X, y_k)\, P(Y = y_k)}$

Theorem 5.29 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$