Claim: X a R.V on $\Omega \to N$ die rolls

$n \le N$. $\mathcal{F}_n \to$ info from first n die rolls.

$E_n X \to$ cond exp of X given $\mathcal{F}_n$

$\boxed{Y \to \mathcal{F}_n \text{ meas.}}$

Claim: $E(X - E_n X)^2 \le E(X - Y)^2$

(Intuition $\longrightarrow$ think of $\left(E(Y-Z)^2\right)^{1/2}$ as a measure of the "distance" between 2 RV's Y & Z

"Amongst all $\mathcal{F}_n$ meas RV's Y, $E_n X$ is the "best approx" to X".

$$\underbrace{E(X-Y)^2}_{\longrightarrow\ E(X - E_n X)^2} \longrightarrow \text{Exp} \longrightarrow ??$$

$$E(X-Y)^2 = E\left((X-E_nX)+(E_nX-Y)\right)^2$$

have shown this is 0 below!

$$= \underbrace{E(X-E_nX)^2}_{\text{What we want}} + \underbrace{E(E_nX-Y)^2}_{\geq 0} + \boxed{2E(X-E_nX)(E_nX-Y)}$$

???

Going to use 2 properties of E_n to evaluate

(class today): (1) Y & Z 2 RV's. $\boxed{Y \text{ is } \mathcal{F}_n \text{ meas.}}$

$\overline{Z} \longrightarrow$ anything $\longleftarrow$

Thm $E_n(YZ) = Y E_n Z$

$E_n Y = \cancel{EY}$ ①

$= Y$ ②

"best approx of Y by an F_n meas RV"

② $E_n(YZ) \overset{?}{=} Y E_n Z$ (FALSE!)

$\neq$ RV $\neq$

$\mathbb{E}$

(2) $Z \to$ any RV. Q: $E_n Z \to \mathcal{F}_n$ meas RV

What is $\mathbb{E}(E_n Z)$? Claim: $\mathbb{E} E_n Z = \mathbb{E} Z$

(1) $E_n Z$ is an $\mathcal{F}_n$ meas RV.

(2) For every $A \subseteq \mathcal{F}_n$, $\sum_{\omega \in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$

Put $A = \Omega$ ($\because \Omega \in \mathcal{F}_n \ \forall n$ incl $n = 0$)

$$\Rightarrow \boxed{E(E_n X) = EX}$$

Term in Box :

$$\boxed{2E(X-E_nX)(E_nX-Y)}$$

$$E\left[(X-E_nX)(E_nX-Y)\right] = E E_n (X-E_nX)(E_nX-Y)$$

F_n - meas

$$= E\left((E_nX-Y)\left(E_n(X-E_nX)\right)\right)$$

$$\underbrace{\left(E_n X - \underbrace{E_n E_n X}_{E_n X}\right)}_{0}$$