Definition 5.24. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of* X *given* $\mathcal{F}_n$, denoted by $\boldsymbol{E}_n X$, or $\boldsymbol{E}(X \mid \mathcal{F}_n)$, to be the unique *random variable* such that:

$E_n X$ = best approx of X as a $\mathcal{F}_n$ meas R.V.

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.
(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

Remark 5.25. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

Risk Neutral pricing formula

Given $V_N \to$ payoff at maturity of some security

AFP at time $n \leq N$? RNP formula

Cond exp under risk Neutral meas. $\longrightarrow$ $V_n = \tilde{E}_n(V_N \text{ (discount factor)})$

Prop:

Remark 5.26 (Uniqueness). If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega\in A} Y(\omega)p(\omega) = \sum_{\omega\in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.

Pf: Let $X = Y - Z$. $\Rightarrow$ X is $\mathcal{F}_n$ - meas & $\forall A \in \mathcal{F}_n$, $\sum_{\omega\in A} X(\omega)p(\omega) = 0$

Let $A = \{X > 0\}$ (Note $A \in \mathcal{F}_n$ $\because$ X is $\mathcal{F}_n$ meas).

Then by assumption $\sum_{\omega\in A} X(\omega)p(\omega) = 0$

Also, $\underbrace{\sum_{\omega\in A} X(\omega)p(\omega)}_{=0} \geq \underbrace{\left(\min_{\omega\in A} X(\omega)\right)}_{>0} \overbrace{\sum_{\omega\in A} p(\omega)}^{P(A)}$

$\swarrow$ only possible if $P(A) = 0$

III^{ly} $B = \{X < 0\}$ $\longrightarrow$ get $P(B) = 0$. $\therefore P(X \neq 0) = 0 \Leftrightarrow P(Y = Z) = 1$

QED.

Theorem 5.27. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$. (On homework).*
(2) If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.

(Tower Property)

Proof: Claim ① $E_m E_n X$ is $\mathcal{F}_m$ - meas

② $\forall A \in \mathcal{F}_m$, $\sum_{\omega \in A} E_m E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$ $\Rightarrow E_m E_n X = E_m X$

$\Rightarrow$ QED

Pf of ① $\longrightarrow$ True because $E_m(\ldots)$ is $\mathcal{F}_m$ - meas.

② $\rightarrow \sum_{\omega \in A} E_m (E_n X(\omega)) p(\omega) = \sum_{\omega \in A} E_n X(\omega) p(\omega)$ $\quad (\because A \in \mathcal{F}_m)$

$= \sum_{\omega \in A} X(\omega) p(\omega)$ $\quad (\because A \in \mathcal{F}_n)$

QED.

(3) If $\boxed{X \text{ is } \mathcal{F}_n \text{ measurable,}}$ and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_nY$.

Pf: Will show ① $X E_n Y$ is an $\mathcal{F}_n$ meas RV

② $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} X(\omega) E_n Y(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) Y(\omega) p(\omega)$

Pf of ① $\to$ True bc X is $\mathcal{F}_n$ meas & $E_n Y$ is $\mathcal{F}_n$-meas

(X takes on vals $x_1 \dots x_k$).

② Pf: $\sum_{\omega \in A} X(\omega) E_n Y(\omega) p(\omega) = \sum_{i=1}^{k} \sum_{\omega \mid X(\omega) = x_i \cap A} X(\omega) E_n Y(\omega) p(\omega)$

$= \sum_{i=1}^{k} x_i \sum_{\omega \in \{X = x_i\} \cap A} E_n Y(\omega) p(\omega).$

$$= \sum_{i=1}^{k} x_i \left(\sum_{\omega \in \{X = x_i\} \cap A} E_n Y(\omega)\, p(\omega) \right)$$

X is $\mathcal{F}_n$-meas.

$\downarrow$

$\left(\because \{X = x_i\} \in \mathcal{F}_n \right)$

$$= \sum_{i=1}^{k} x_i \left(\sum_{\omega \in \{X = x_i\} \cap A} Y(\omega)\, p(\omega) \right)$$

$$= \sum_{i=1}^{k} \sum_{\omega \in \{X = x_i\} \cap A} x_i Y(\omega)\, p(\omega)$$

$$= \sum_{i=1}^{k} \sum_{\omega \in \{X = x_i\} \cap A} X(\omega) Y(\omega)\, p(\omega) = \sum_{\omega \in A} X(\omega) Y(\omega)\, p(\omega)$$

QED.

Theorem 5.28. *If X is independent of $\mathcal{F}_n$ then* $\boxed{\boldsymbol{E}_n X = \boldsymbol{E} X}$.

$E_n X$

$E(X \mid \mathcal{F}_n)$

$P(X \mid A)$

ent

Q6 on HW

$$E(X-Y)^2 \geq E(X - E_n X)^2 \quad \forall\ \mathcal{F}_n\text{-meas } Y$$

Hint $E(X-Y)^2 = E\left(\underbrace{(X - E_n X)} + \underbrace{(E_n X - Y)}\right)^2$ & expand.