Market $n-1$ assets S^1 --- S^{n-1} & money market

Harrison..

→ $\Omega = \{\omega_1, \dots \omega_m\}$

$S^1(\omega_i) = $ ______

Conditions for ① When is this market complete?

② " " " " arb free?

last time: A_1 A_2 are ind if $P(A_1 \cap A_2) = P(A_1)P(A_2)$

$\hookrightarrow$ $A_1 \dots A_n$ is ind if $P(A_{i_1} \cap \dots A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k})$

$\forall$ sub collections $\{A_{i_1}, \dots A_{i_k}\} \subseteq \{A_1, \dots A_n\}$.

$\{X = x\}$ ~~=~~ short hand for $\{\omega \in \Omega \mid X(\omega) = x\}$

$$\underline{P(\underline{X}=x)} = \underline{P}\left(\{\omega \in \Omega \mid X(\omega) = x\}\right)$$

Definition 5.11. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Question 5.12. *What does it mean for the random variables $X_1, \ldots, X_n$ to be independent?*

Question 5.13. *Are uncorrelated random variables independent?*

$$\forall x_1 \cdots x_n \in \mathbb{R} \quad P(X_1 = x_1,\ X_2 = x_2 \cdots X_n = x_n) = P(X_1 = x_1) P(X_2 = x_2) \cdots P(X_n = x_n)$$

looks like not enough.

need $P(X_{i_1} = x_{i_1}, \cdots X_{i_k} = x_{i_k}) = P(X_{i_1} = x_{i_1}) \cdots P(X_{i_k} = x_{i_k})$

$\forall$ sub coll

Turns out is enough

Theorem 5.14. *The random variables $X_1, \ldots, X_n$ are independent if and only if for all $x_1, \ldots, x_n \in \mathbb{R}$ we have*

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2)\cdots\boldsymbol{P}(X_n = x_n)\,. \quad (\forall x_1, \ldots, x_n \in \mathbb{R})$$

Corollary 5.15. *Suppose for simplicity all coin tosses have M outcomes (i.e. $\omega_i \in \{1, \ldots, M\}$). Let p be a probability mass function. The coin tosses are all independent, if and only if, there exists functions $p_1, \ldots, p_N$ such that $p(\omega) = p_1(\omega_1)p_2(\omega_2)\cdots p_N(\omega_N)$.*

Check when $n = 3$. Say X, Y, Z are ind. $\Rightarrow \forall x, y, z$

$$P(X = x, Y = y, Z = z) = P(X = x)P(Y = y)P(Z = x).$$

$\Rightarrow$ forward direction.

Converse: Say $\forall x, y, z$, $P(X = x, Y = y, Z = z) = P(X = x)P(Y = y)P(Z = z)$.

Then $\forall x, y \in \mathbb{R}$, $P(X = x, Y = y) = P(X = x)P(Y = y)$

(& similarly for X, Z or Y, Z).

Pf: Say Z takes on the values $z_1, \ldots, z_n$.

$$P(X=x, Y=y) = \sum_{i=1}^{n} P(X=x, Y=y, Z=z_i)$$

$$\stackrel{\text{assumption}}{=} \sum_{i=1}^{n} P(X=x)\,P(Y=y)\,P(Z=z_i)$$

$$= P(X=x)\,P(Y=y) \underbrace{\sum_{i=1}^{n} P(Z=z_i)}_{1}$$

QED

General case: Induction. (see std book)

5.2. **Filtrations and adapted processes.**

(Binomial model: $d_i = 2\ \forall i$)

- Let $N \in \mathbb{N}$, $d_1, \dots, d_N \in \mathbb{N}$, $\Omega = \{1, \dots, d_1\} \times \{1, \dots, d_2\} \times \cdots \times \{1, \dots, d_N\}$.
- That is $\Omega = \{\omega \mid \omega = (\omega_1, \dots, \omega_N),\ \omega_i \in \{1, \dots, d_i\}\}$.
- $d_n = 2$ for all n corresponds to flipping a two sided coin at every time step.

Definition 5.16. We define a *filtration* on Ω as follows: (information)

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

▷ $\mathcal{F}_1$ = all events that can be described by only the first coin toss (die roll). E.g. $A = \{\omega \mid \omega_1 = H\} \in \mathcal{F}_1$.

▷ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses. (die rolls)

Question 5.17. *Let* $\Omega = \{H, T\}^3 \cong \{1, 2\}^3$. *What are* $\mathcal{F}_0, \dots, \mathcal{F}_3$*?*

$\mathcal{F}_0$ = info before rolling any die.

$A_1 = \{\omega \in \Omega \mid \omega = (\omega_1 \dots \omega_N)\ \&\ \omega_1 = 1\} \in \mathcal{F}_1$

$A_2 = \{\omega \in \Omega \mid \omega = (\quad)\ \&\ \omega_2 = 1\} \notin \mathcal{F}_1$

Claim: # events in $\mathcal{F}_1 = 2^{d_1}$

$A \subseteq \underbrace{\{1, \dots d_1\}}$ $\qquad \left(2^{d_1} \text{ choices for } A\right)^2$

$\underbrace{A \times \{1, \dots d_2\} \times \{1, \dots d_3\} \times \dots \{1, \dots d_N\}} \in \mathcal{F}_1$

$\mathcal{F}_2 = \{ \underbrace{A}_{2^{d_1}} \times \underbrace{B}_{2^{d_2}} \times \{1, \dots d_3\} \times \dots \{1, \dots d_N\} \}$

$N = 3$, $\Omega = \underline{\{\pm 1\}^3} = \{\omega \mid \omega = (\omega_1, \omega_2, \omega_3)\ \&\ \omega_i \in \{\pm 1\}\}$.

Enumerate $\underline{\mathcal{F}_0}$, $\mathcal{F}_1$, $\mathcal{F}_2$ & $\mathcal{F}_3$

$\mathcal{F}_0 = \{\phi, \Omega\}$.

$\mathcal{F}_1 = \{\phi, \Omega, \{(1,1,1), (1,-1,1), (1,1,-1), (1,-1,-1)\}$

$, \{(-1,1,1), (-1\ \ 1\ \ \ (-1\ 1\ \ \ \ (-1\ \ 1\ \ )\}.$

$\mathcal{F}_2 = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \{(1,1,1), (1,1,-1)\} \cup \ldots$

Note $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$

power set.

Definition 5.18. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \ldots, \omega_n$.

▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.

Question 5.19. *Let* $X(\omega) \stackrel{\text{def}}{=} \omega_1 - 10\omega_2$. *For what n is $\mathcal{F}_n$-measurable?*

$\{X \in [0,1]\}$

$X(\omega) = \omega_1 - 10\omega_2$

Is X $\mathcal{F}_0$-measurable? NO

" " $\mathcal{F}_1$ - " ? NO

" " $\mathcal{F}_2$ - " ? YES

" " $\mathcal{F}_3$ " ? YES.

$\Omega = \{-1, 0, 1\}^2$

$\{(1,0), (1,1), (1,-1)\}$

$\mathcal{F}_1 = 8$ elmts

$\emptyset \in \mathcal{F}_1$, $\Omega \in \mathcal{F}_1$, $\{(0,0), (0,1), (0,-1)\} \in \mathcal{F}_1$

$\{(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1)\} \in \mathcal{F}_1$

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$\boxed{\text{1st roll} \to 0 \text{ or } 1}$

2nd roll $\to$ anything