Q2a: $0 < f_1 < f_2 < f_3$ $\qquad r > -1$

Q: When is this arb free

$$S_0 \nearrow \rightarrow \searrow \left.\begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix}\right\} \leftarrow S_1$$

Guess $f_1 < 1+r < f_2$

Reason: $\longrightarrow$ Write pf

$$X_0 \begin{cases} \to \text{Cash} \\ \to \text{Stock} \quad (\Delta_0 \text{ shares}). \end{cases}$$

$$X_1 = (1+r)\left(X_0 - \Delta_0 S_0\right) + \Delta_0 S_1$$

$$= \underbrace{(1+r)X_0}_{\text{Start with } 0} + \Delta_0\big(S_1 - (1+r)S_0\big)$$

Assume $X_1 \geq 0 \iff \Delta_0\big(S_1 - (1+r)S_0\big) \geq 0$

No arb $\iff$ If $X_0 = 0$ & $\underline{X_1 \geq 0}$ $\Rightarrow X_1 = 0$

① If $S_1 - (1+r)S_0$ is always ≥ 0 $\longrightarrow$ have arb (∵ choose $\Delta_0 > 0$)

② " " " " ≤ 0 $\longrightarrow$ " ($\Delta_0 < 0$)

③ No arb $\iff$ $S_1 - (1+r) S_0$ is not const sign.

$\iff f_1 < 1+r < f_3$

Q2b Mkt has no arb. find $\tilde{p}_i \in (0,1)$ +

① $\sum \tilde{p}_i = 1$ & ② $\sum \tilde{p}_i f_i = (1+r)$ ③ $\tilde{p}_2 = \varepsilon$

$$\tilde{p}_1 f_1 + \varepsilon f_2 + (1 - \tilde{p}_1 - \varepsilon) f_3 = 1+r$$

$$\iff \tilde{p}_1 (f_1 - f_3) = 1+r - (1-\varepsilon) f_3 - \varepsilon f_2$$

$$\Leftrightarrow \tilde{f}_1 = \frac{(1-\varepsilon)f_3 + \varepsilon f_2 - (1+r)}{f_3 - f_1}$$

Want $\in (0, 1)$

Can choose ε. Try an arrange $\tilde{f}_1 > 0$

Know $0 < f_1 < f_2 < f_3$

and $f_1 < 1+r < f_3$

$$f_3 - (1+r) > 0$$

Want $f_3 - (1+r) > \varepsilon(\underbrace{f_3 - f_2}) \Leftrightarrow \varepsilon < \boxed{\dfrac{f_3 - (1+r)}{f_3 - f_2}}$

Choose any $\varepsilon \in (\quad , \quad)$

& write $\tilde{q}_1, \ldots \tilde{q}_3$ in terms of a_n

Q2 f → $f_1 = \frac{1}{2}$, $f_2 = 1$, $f_3 = 2$, $r > 0$

$$V_1 = (S_1 - 1)^+ \qquad S_0 = 1$$

$$V_1 = \begin{cases} 1 & \text{if roll } 3 \\ 0 & \text{if roll } 1, 2. \end{cases}$$

Find V_0. ⟶ Q2e → Can't replicate.

Remark: ① If you can replicate $\longrightarrow$ AFP is unique & = initial capital.

② " " CAN'T " $\longrightarrow$ " " NOT "

$\hookrightarrow$ find all poss values for V_0. $\longrightarrow$ find $\underbrace{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3}$ $\longrightarrow$

Know:

$$ES_1 = (1+r) S_0 \quad \& \quad \sum \tilde{p}_i = 1$$

Given Extended market is arb free!

Start with $X_0 = 0$

$\to \Delta_0$ shares of Stock.

$\to \Gamma_0$ options (valued at V_0)

$\to -\Delta_0 S_0 - \Gamma_0 V_0$ Cash

Then $X_1 =$ wealth at time $1 = (1+r)(-\Delta_0 S_0 - \Gamma_0 V_0) + \Delta_0 S_1 + \Gamma_0 V_1$

need market to remain arb free

i.e. If $X_1 \geq 0$ ($\forall$ cells) then $X_1 = 0$

$$X_1 = \Delta_0\left(S_1 - (1+r)S_0\right) + \Gamma_0\left(V_1 - (1+r)V_0\right)$$

world 1: $X_1 = \Delta_0\left(\frac{1}{2} - (1+r)\right) - \Gamma_0(1+r)V_0 \geq 0$

world 2: $X_1 = \Delta_0\left(1 - (1+r)\right) - \Gamma_0(1+r)V_0 \geq 0$

world 3: $X_1 = \Delta_0\left(2 - (1+r)\right) + \Gamma_0\left(1 - (1+r)V_0\right) \geq 0$

Can choose $\underline{\Delta_0}$ & $\underline{\Gamma_0}$ arbitrarily.