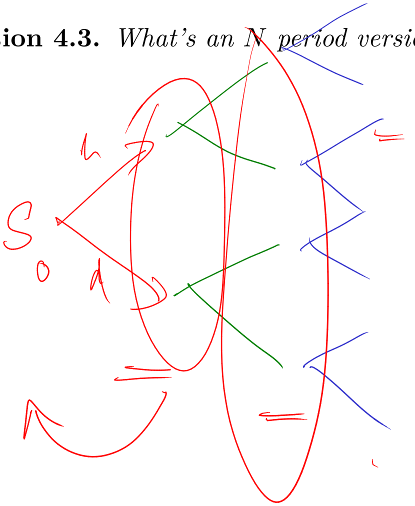
Question 4.3. *What's an N period version of this model? Do we have the same formulae?*



Q1 No arb ? $(d < 1+r < u)$

Q2: Price sec

Q3: American options

Can exercise at any time

Notation $\omega = (\omega_1, \omega_2, \cdots \omega_N) \in \Omega$

5. Probability spaces, in our context

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 5.1. The *Sample space* is the set $\Omega = \{(\omega_1, \ldots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a } \cancel{\text{coin}} \text{ toss.}\}$ dice roll

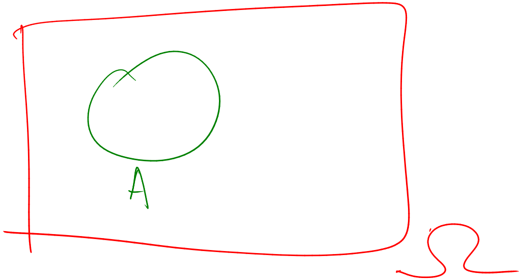
- ▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$.
- ▷ Coins don't have to be identical: Pick $M_1, M_2, \ldots, \in \mathbb{N}$, and can require $\omega_i \in \{1, \ldots, M_i\}$.

Definition 5.2. A *sample point* is a point $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$.

Definition 5.3. A *probability mass function* is a function $p \colon \Omega \to [0, 1]$ such that $\boxed{\sum_{\omega \in \Omega} p(\omega) = 1.}$

Definition 5.4. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

$p(\omega)$ = Prob of "ω" occurring.



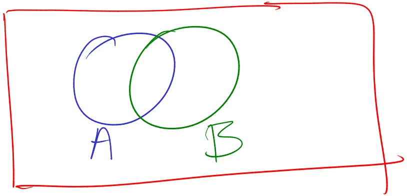
$A \subset \Omega$ (event)

$$P(A) = \sum_{\omega \in A} p(\omega)$$

5.1. **Independence.**

Definition 5.5. Two events are independent if $\boxed{\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)}$. (multiplication law)

Question 5.6. *What does it mean for the events $A_1, \ldots, A_n$ to be independent?*



Say A occurs.

Q: Prob B occurs given A occurred

$$\hookrightarrow P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Find: $P(A) = P(A|B)$

$$P(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} \iff \boxed{P(A \cap B) = P(A)P(B)}$$

Def : A_1, A_2, A_3 are ind if $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

AND $P(A_1 \cap A_2) = P(A_1)P(A_2)$ & $P(A_2 \cap A_3) = P(A_2)P(A_3)$

$P(A_1 \cap A_3) = P(A_1)P(A_3)$

Def : $A_1, \dots A_N$ are ind if for any sub collection $\{A_{i_1}, A_{i_2}, \dots, A_{i_m}\}$, we have $P\left(\bigcap_{j=1}^{m} A_{i_j}\right) = \prod_{j=1}^{m} P(A_{i_j})$

Definition 5.7. A *random variable* is a function $X : \Omega \to \mathbb{R}$. (Later will allow $X : \Omega \to \mathbb{R}^d$).

Question 5.8. *What is the random variable corresponding to the outcome of the n^{th} coin toss?*

$\Omega = \{1, 2, \dots 6\}^N$ $(N \in \mathbb{N})$ $\omega = (\omega_1, \dots \omega_N)$ & each $\omega_i \in \{1, \dots 6\}$.

X_2 = RV corresponding to roll of the second die.

Eg: $\boxed{X_2(\omega) = \omega_2}$

(here $\omega = (\omega_1, \omega_2, \dots \omega_N)$)

ω

ω

Ω

Definition 5.9. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum x_i \boldsymbol{P}(X = x_i)$.

Definition 5.10. The *variance* is $\boldsymbol{E}(X - \boldsymbol{E}X)^2 = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

X is a R.V. $(X : \Omega \longrightarrow \mathbb{R})$

EX = average value of X $= \sum_{\omega \in \Omega} X(\omega)\, p(\omega) = \sum_{i=1}^{M} x_i P(X = x_i)$

Ω finite $\Rightarrow$ $X(\Omega)$ = range of X is finite $= \{x_1, x_2, \ldots x_M\}$.

$\{\omega \in \Omega \mid X(\omega) = x_i\} \subseteq \Omega$ (some event)

Notation: $\{X = x_i\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x_i\} = X^{-1}(x_i)$

Definition 5.11. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Question 5.12. *What does it mean for the random variables $X_1, \ldots, X_n$ to be independent?*

Question 5.13. *Are uncorrelated random variables independent?*

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

$$\{Y = y\} = \{\omega \in \Omega \mid Y(\omega) = y\}$$

are ind

not enough

$$P(X_1 = x_1 \,\&\, X_2 = x_2 \cdots X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i) \quad \forall x_i$$

Def: $\forall x_i$, the events $\{X_1 = x_1\}, \{X_2 = x_2\} \ldots \{X_n = x_n\}$ are all ind

(Then $P(X_1 = x_1 \,\&\, X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2)$ etc)