Last time:

1. Arbitrage $\longrightarrow X_0$ (initial wealth) $= 0$

No arbitrage means $\underbrace{X_n}_{\text{wealth at time } n} \geq 0$ then $\boxed{X_n = 0}$

2. AFP: Traded assets (& M.M.)

Security (NTA) $\longrightarrow$ AFP $= V_0$ if given the option to trade the NTA at price V_0, the market remains arb free.

3. Pricing = Replication!

Question 3.4. *Consider a financial market with a money market account with interest rate r, and a stock. Let $K > 0$. A* forward contract *requires the holder to buy the stock at price K at maturity time N. What is the arbitrage free price at time 0?*

Forward Contract: Req to buy at price K at time N.

① Money Market (r)
② Stock.

Q: Compute AFP (time 0).

price at time N

Q: Payoff at maturity? ⟶ $S_N - K$

$S_N - K$

Q: Replicate ⟶ ① buy 1 share of stock
② short $\frac{K}{(1+r)^N}$ cash

$\underbrace{\qquad\qquad\qquad}_{\text{time } 0}$

① 1 stock
② K cash (short).

$\underbrace{\qquad\qquad\qquad}_{\text{time } N}$

$$AFP = S_0 - \frac{K}{(1+R)^N}$$

4. Binomial model (one period)

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Say we have access to a money market account with interest rate r. The *binomial model* dictates that the stock price varies as follows. Let $p \in (0,1)$, $q = 1-p$, $0 < d < u$ (up and down factors). Flip a coin that lands heads with probability p, and tails with probability q. When the coin lands heads, the stock price changes by the factor u, and when it lands tails it changes by the factor p.

Question 4.1. *When is there arbitrage in this market?*

No arb $\iff d < 1+r < u$

$S_0 \xrightarrow{p\ (H)} uS_0$ (up factor u)

$S_0 \xrightarrow{q\ (T)} dS_0$ (down factor)

(time 0) (time 1)

X_0 = Initial wealth $\to$ Δ_0 shares of stock

$X_0 - \Delta_0 S_0 \to$ cash.

X_1 = wealth at time 1

$$= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$$

$$= (1+r)X_0 + (1+r)\Delta_0\left(\frac{S_1}{1+r} - S_0\right)$$

No arb: $X_0 = 0$

$$X_1 = \Delta_0\left(S_1 - (1+r)S_0\right)$$

Question 4.2. *If a security pays V_1 at time 1, what is the arbitrage free price at time 0. (V_1 can depend on whether the coin flip is heads or tails).*

Get $V_1(H)$ for heads & $V_1(T)$ for tails. ①

Wealth if heads $= X_1(H) = (1+r)X_0 + (1+r)\Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right)$ Want $= V_1(H)$

" " tails $= X_1(T) = (1+r)X_0 + (1+r)\Delta_0\left(\frac{S_1(T)}{1+r} - S_0\right)$ ② Want $= V_1(T)$

Solve ↗ Find $\tilde p, \tilde q$ $\ni$ $\tilde p + \tilde q = 1$ & $\frac{\tilde p S_1(H) + \tilde q S_1(T)}{1+r} = \cancel{(1+r)} S_0$

①$\tilde p$ + ②$\tilde q$ $\Rightarrow$ $\boxed{\frac{\tilde p V_1(H) + \tilde q (V_1(T))}{1+r} = X_0} + 0$

$\boxed{X_0 = \tilde E \frac{V_1}{(1+r)}}$ ← RNP

AFP.

Δ_0 : ① − ② $\Rightarrow$ $\frac{V_1(H) - V_1(T)}{\cancel{1+r}} = \Delta_0\left(\frac{S_1(H)}{\cancel{1+r}} - \frac{S_1(T)}{\cancel{1+r}}\right)$

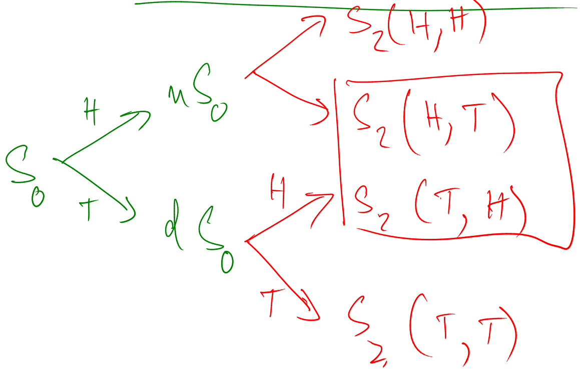
find $\tilde{p}, \tilde{q}$: $\tilde{p}\, S_1(H) + \tilde{q}\, S_1(T) = (1+r)\, S_0$

$\iff \tilde{p}\, u \cancel{S_0} + (1-\tilde{p})\, d \cancel{S_0} = (1+r) \cancel{S_0}$

$\tilde{p}(u-d) + d = 1+r \Rightarrow \boxed{\tilde{p} = \dfrac{1+r-d}{u-d}}$

Note $d < 1+r < u \iff \tilde{p} \in (0,1)$ (can be int as a prob).

Question 4.3. *What's an N period version of this model? Do we have the same formulae?*



No arb?

Guess: $d < (1+r) < u$

Q2!

Unknowns? $\longrightarrow$ 2 X_0, Δ_0

Δ_1 (2 unks)

$\Delta_2 \to 4$

2^N outcomes

2^N eq..