Math 370 Homework.

Please be aware of the late homework, and academic integrity policies in the syllabus. In particular, you may collaborate, but must write up solutions on your own. You may only turn in solutions you understand.

Assignment 1 (assigned 2020-09-02, due 2020-09-09).

1. Consider a market which only trades a stock and a forward contracts with delivery time 1. The initial price of the stock is $S_0 = \$10$, the forward price $F = \$12$, and at time 1 the stock price S_1 only takes two values: \$11 or \$14. In this market show how you can replicate a call option on the stock with strike price \$13 and maturity time 1. Use this to compute the arbitrage free price of the call.

   Note: A *forward contract* with delivery price K (and delivery time 1) pays $S_1 - K$ at time 1. The *forward price* is that value of K which makes the price of the forward contract \$0 initially.

2. Let $0 < f_1 < f_2 < f_3$, and $r > -1$. Consider a financial market with a stock and a money market account. The money market has interest rate $r > -1$. The stock price changes according to the roll of a fair 3 sided die. If the die rolls $i \in \{1, \dots, 3\}$, then $S_1 = f_i S_0$. (Here S_0 and S_1 are the stock prices at time 1 and time 0 respectively.)
   (a) Find necessary and sufficient conditions on f_1, f_2 and f_3 under which the market has no arbitrage.
   (b) Assuming the market has no arbitrage, find $\tilde{p}_1, \dots \tilde{p}_3 \in (0,1)$ such that
   $$\sum_{i=1}^{3} \tilde{p}_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{3} \tilde{p}_i f_i = (1+r)\,?$$
   (c) Are the numbers $\tilde{p}_i$ in the previous part unique? Prove it, or find more than one such triple of such numbers.
   (d) Suppose now $S_0 = \$1$, $f_1 = 1/2$, $f_2 = 1$ and $f_3 = 2$, and consider a security that pays (at time 1) \$1 if the die rolls 1, \$2 if the die rolls 2, and \$4 if the die rolls 3. Can you replicate this security? If yes, find the (unique) arbitrage free price. If no, is there at least one price at which the security can be traded so that the extended market still has no arbitrage? (In either case prove your answer.)
   (e) Let S_0, f_i be as in the previous part, suppose $r > 0$, and consider a call option on the stock with strike price \$1. (This option would pay \$1 if the die rolls 3, and \$0 otherwise.) Can you replicate this security?
   (f) For the call option in the previous part, find all $V_0 \geqslant 0$ such that introducing this option into the market at price V_0 keeps the market arbitrage free.

Assignment 2 (assigned 2020-09-09, due 2020-09-16).

1. If A, B, C are three events such that $\boldsymbol{P}(A \cap B \cap C) = \boldsymbol{P}(A)\boldsymbol{P}(B)\boldsymbol{P}(C)$, the events A, B are independent, and the events B, C are independent, then must A, B, C be independent? Prove it, or find a counter example.
2. (a) Suppose $X_1, \dots, X_n$ are n independent random variables. True or false: $\operatorname{Var}(\sum_1^n X_i) = \sum_1^n \operatorname{Var}(X_i)$. Prove it, or find a counter example.
   (b) Conversely, if $\operatorname{Var}(\sum_1^n X_i) = \sum_1^n \operatorname{Var}(X_i)$, must $X_1, \dots, X_n$ be independent? Prove it, or find a counter example.
3. *(Jensen's inequality)* Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a convex function, and X be a random variable.
   (a) Show that $\varphi(\boldsymbol{E}X) \leqslant \boldsymbol{E}\varphi(X)$. [Hint: Convex functions lie above their tangents.]
   (b) If φ is strictly convex, then show that equality holds in the previous part if and only if X is constant.
   (c) Let $p \geqslant 1$, and X be a random variable. Show that $\boldsymbol{E}|X| \leqslant (\boldsymbol{E}|X|^p)^{1/p}$.
   (d) What happens to the previous part if $p \in (0,1)$?

We will subsequently assume Ω is the probability space corresponding to N die rolls (or coin tosses), and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N$ be the filtration corresponding to the die rolls. Given a random variable X, we use the notation $\boldsymbol{E}_n X = \boldsymbol{E}(X \mid \mathcal{F}_n)$ to denote the conditional expectation of X given $\mathcal{F}_n$.

4. Suppose $A, B \in \mathcal{F}_n$. Must $A \cup B \in \mathcal{F}_n$? Prove it. Also, state (without proof) whether $A \cap B$ and A^c must also necessarily belong to $\mathcal{F}_n$?
5. Let X, Y be a random variables.
   (a) For any $\alpha \in \mathbb{R}$ show that $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$ almost surely.
   (b) If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is convex, must $\varphi(\boldsymbol{E}_n X) \leqslant \boldsymbol{E}_n \varphi(X)$ almost surely? Prove it, or find a counter example.
6. *(Least square approximation)* Let X be a random variable, and Y be any $\mathcal{F}_n$-measurable random variable, show that
   $$\boldsymbol{E}(X - Y)^2 \geqslant \boldsymbol{E}(X - \boldsymbol{E}_n X)^2 .$$

Assignment 3 (assigned 2020-09-16, due 2020-09-23).

1. Let Ω be a probability space corresponding to N independent coin tosses, where each coin shows heads with probability $p \in (0,1)$ and tails with probability $q = 1 - p$. Let $\omega = (\omega_1, \dots, \omega_N) \in \Omega$, with each $\omega_i \in \{\pm 1\}$ representing the outcome of the i^{th} coin toss. Let $u, d, S_0 > 0$, and define $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$ and $S_{n+1}(\omega) = dS_n(\omega)$ otherwise.
   (a) Find $\boldsymbol{E}_n S_N$. Express your answer in terms of d, u, p, q, n, N and $S_0, \dots, S_n$.
   (b) Let $r > -1$. Find a necessary and sufficient condition on p, u, d, r such that $\boldsymbol{E}_n S_{n+1} = (1+r)S_n$.
2. Let X, Y be two random variables such that $\boldsymbol{E}X = \boldsymbol{E}Y = 0$, X is independent of $\mathcal{F}_n$, and Y is $\mathcal{F}_n$-measurable. Let $Z = XY$. Show $\boldsymbol{E}_n Z = 0$. Moreover, show by example, that Z need not be independent of $\mathcal{F}_n$.
3. *(Independence Lemma)* If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function then show
$$\boldsymbol{E}_n f(X,Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \text{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$
4. Let Ω be a probability space corresponding to N independent fair coin tosses. Let X_n be the random variable corresponding to the n^{th} coin toss, with $X_n = 1$ corresponding to the coin landing heads and $X_n = -1$ corresponding to the coin landing tails.
   (a) Let $M_n = \sum_{j=1}^{n} X_j$. Is M a martingale? Compute $\boldsymbol{E}M_n$ and $\boldsymbol{E}M_n^2$.
   (b) Let $C, \sigma > 0$ and define $S_n = C^n e^{\sigma M_n}$. Find a relation between C and σ so that S is a martingale.
5. Let X_n be as in the previous problem. Let $a, b \in \mathbb{Z}$, $Y_0 \in (a,b) \cap \mathbb{Z}$ and define $Y_{n+1} = Y_n + X_{n+1}$ if $Y_n \in (a,b)$ and $Y_{n+1} = Y_n$ if $Y_n \in \{a, b\}$.
   (a) Compute $\boldsymbol{E}_n Y_{n+1}$ in terms of $Y_0, \dots, Y_n$.
   (b) Suppose now $a = 0$, $b = 3$, and let $p_n = \boldsymbol{P}(Y_n = 0)$ if $Y_0 = 1$, and let $q_n = \boldsymbol{P}(Y_n = 0)$ if $Y_0 = 2$. Find a relation between p_{n+2} and p_n. Also find a relation between q_{n+2} and q_n.
   (c) Assuming $\lim_{n\to\infty} p_n$ and $\lim_{n\to\infty} q_n$ exist, find $\lim_{n\to\infty} p_n$ and $\lim_{n\to\infty} q_n$.
   (d) *(Optional)* Find p_n and q_n. [We will encounter and solve many such recurrence relations later in the course, and it may be helpful to try your had at it now.]

Assignment 4 (assigned 2020-09-23, due Never).

In light of your midterm 2020-09-30, this homework is optional. These are good problems to use for review and practice, and some of them will be on your next homework.

1. Consider the multi-period binomial model with $u = 2$, $d = 1/2$ and $r = 1/4$.
   (a) What is the distribution of S_3 under the risk neutral measure?
   (b) Compute $\tilde{\boldsymbol{E}}S_1$, $\tilde{\boldsymbol{E}}S_2$ and $\tilde{\boldsymbol{E}}S_3$. What is the average growth rate of S under $\tilde{\boldsymbol{P}}$?
   (c) Suppose that the actual probabilities of the coin landing heads and tails are $2/3$ and $1/3$ respectively. Redo the previous two parts under the actual probability measure $\boldsymbol{P}$ instead of the risk neutral measure $\tilde{\boldsymbol{P}}$.
2. Let $\Omega = \{-1, 1\}^N$, $p_1 \in (0,1)$, $q_1 = 1 - p_1$, and let $\boldsymbol{P}$ be the probability measure on Ω under which the outcome of each coin oss $\omega_1, \dots, \omega_N$ are i.i.d. with $\boldsymbol{P}(\omega_i = 1) = p_1$. Let u_n, r_n, d_n be adapted processes such that $d_n < 1 + r_n < u_n$, and $r_n > -1$ almost surely. Let $S_0 > 0$, $D_0 = 1$ and define the process D_n and S_n by
$$D_{n+1}(\omega) = (1 + r_n(\omega))^{-1} D_n(\omega)\,, \qquad S_{n+1}(\omega) = \begin{cases} u_n(\omega)S_n(\omega) & \text{if } \omega_{n+1} = 1\,, \\ d_n(\omega)S_n(\omega) & \text{if } \omega_{n+1} = -1\,. \end{cases}$$
   (a) Are D_n and S_n adapted? Are they predictable? Prove it. (A process X is called predictable if for all n, X_{n+1} is $\mathcal{F}_n$ measurable.)
   (b) Is there a measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale? Prove it.
   (c) Must the coin tosses $\omega_1, \dots, \omega_N$ independent under $\tilde{\boldsymbol{P}}$? Justify.
   (d) Must the coin tosses $\omega_1, \dots, \omega_N$ be identically distributed under $\tilde{\boldsymbol{P}}$? Justify.

   Now consider a market with a stock whose price is modelled by S_n, and a bank with random interest rate r_n (i.e.\$1 cash in the bank at time n becomes $\$(1 + r_n)$ cash in the bank at time $n + 1$).
   (e) Show that the process X is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.
   (f) Can there be arbitrage in this market? Prove it.
   (g) Let V_N be an $\mathcal{F}_N$ measurable random variable, and consider a security that pays V_N at maturity. Show that the arbitrage free price of this security at time $n \leqslant N$ is given by $V_n = \tilde{\boldsymbol{E}}_n(D_N V_N / D_n)$.
3. Consider a financial market with a money market account (with interest rate $r = 0$) and a stock with initial price \$100. At time $n + 1$ the stock price either increases by \$10, or decreases by \$10 based on a coin toss. Consider an European call on this stock with strike \$80 and maturity time 5. What is the price of this call at time 0?

Assignment 5 (assigned 2020-09-30, due 2020-10-07).

1. Do questions 2 and 3 from homework 4.
2. Consider the N-period binomial model with parameters u, d, r such that $0 < d < 1 + r < u$. Let $\hat\omega \in \Omega$. The *digital option* (also called the Arrow-Debreu security) pays \$1 if the sequence of coin tosses exactly matches $\hat\omega$, and nothing otherwise. Find the arbitrage free price of this security at each time $n \leqslant N$. Also find the number of shares of stock held in the replicating portfolio at each time $n \leqslant N$.

Assignment 6 (assigned 2020-10-07, due 2020-10-14).

1. Let p be a strictly positive probability mass function on Ω under which the die rolls are not necessarily independent. Let $\hat\omega = (\hat\omega_1, \dots, \hat\omega_N) \in \Omega$ represent a fixed sequence of die rolls, and let X be a $\mathcal{F}_N$-measurable random variable whose range is $x_1, \dots, x_M$. Show that

$$\boldsymbol{E}_n X(\hat\omega) = \sum_{i=1}^{M} x_i \boldsymbol{P}(X = x_i \mid \omega_1 = \hat\omega_1, \dots, \omega_n = \hat\omega_n)\,.$$

[While this looks like an "explicit" formula for conditional expectation, it doesn't work in more general settings. This is why we have avoided using (or even mentioning) this formula so far.]

2. Let $X_1, X_2, \dots, X_N$ be i.i.d. fair coin tosses (1 is heads, -1 is tails).
   (a) Define $S_n = \sum_1^n X_k$. Is S_n^3 always a martingale? Is S_n^3 always a Markov process? Prove your answers to both.
   (b) Define $M_0 = 0$ and $M_n = \sum_0^{n-1} S_k X_{k+1}$. Is M always a martingale? ~~Is M always a Markov process?~~ Prove it.
3. Consider an N period binomial model with $0 < d < 1 + r < u$. An Asian option has payoff of the form $V_N = f(\sum_0^N S_n / N)$ for some (non-random) function f. For instance, $f(x) = (x - K)^+$ corresponds to an Asian call option with strike K.
   (a) Let $Y_n = \sum_{k=0}^n S_k$. Show that Y need not be a Markov process with respect to the risk neutral measure.
   (b) Let $X_n = (S_n, Y_n)$. Show that X is a Markov process with respect to the risk neutral measure.
   (c) Consider a security that pays $f(Y_N)$ at time N. Can the arbitrage free price of this security at time n be expressed in the form $f_n(S_n, Y_n)$? If yes, provide a (recursive) algorithm for computing f_n.
4. Consider an N period binomial model with $0 < d < 1 + r < u$. A *variance swap* that expires at time N pays

$$V_N = \frac{1}{N} \sum_{n=1}^{N} \Big(\log\Big(\frac{S_n}{S_{n-1}}\Big)\Big)^2 - K^2\,,$$

where S_n is the stock price at time n, and K is the strike price. Find a formula for K in terms of the model parameters, such that it costs nothing to enter this contract at time 0.

Assignment 7 (assigned 2020-10-14, due 2020-10-21).

1. Consider the N period binomial model with $0 < d < 1 + r < u$.
   (a) Let τ be a finite stopping time and X is the wealth of a self-financing portfolio with $X_0 = 0$ and $X_\tau \geqslant 0$. Let Δ_n be the number of shares of stock held by the portfolio at time n. Define $\Gamma_n = \Delta_n \mathbf{1}_{\{n<\tau\}}$, and consider a new self-financing portfolio Y with initial wealth $Y_0 = 0$ that holds Γ_n shares of stock at time n. Express Y_N in terms of X_τ, and use this to show $X_\tau = 0$.
   [A simpler way to directly show $X_\tau = 0$ is using the optional sampling theorem.]
   (b) Suppose now that $\tau\colon \Omega \to \{0, \dots, N\}$ is a finite (random) time, and not necessarily a stopping time. Does your proof from the previous part still work? If yes, explain. If no, explain and also find an example where $X_\tau \geqslant 0$, but is not identically 0.
2. Consider the N period binomial model with $0 < d < 1 + r < u$. The *down and rebate* option with face value F and lower barrier L pays F at the first time the stock price is below (or equal) to L. If the stock price never crosses this barrier, then the option expires worthless.
   (a) Write down an algorithm/formula to price this option and find the trading strategy required to replicate this option.
   (b) Using your algorithm above and a computer, compute the price of this option at time 0 with $N = 100$, $u = 1.1$, $d = 0.9$, $r = 1\%$, $S_0 = \$10$, $F = \$1$, $L = \$8.50$. You should also submit the listing of a program that implements this algorithm in your language of choice. (I have uploaded two Python programs pricing *up and rebate* options. You might find it helpful to look at these and modify them as appropriate / port them to your language of choice.)
3. Consider the N period binomial model with $0 < d < 1 + r < u$, and denote the stock price by S_n. Consider an *up and out* option with strike K and barrier price U. If the stock price ever exceeds (or equals) U, the option expires worthless. If not, the option pays $(S_N - K)^+$ at maturity time N.
   (a) Write down an algorithm/formula to price this option and find the trading strategy required to replicate this option.
   (b) Using your algorithm above and a computer, compute the price of this option at time 0 with $N = 90$, $u = 1.05$, $d = 0.9$, $r = 3\%$, $S_0 = \$10$, $K = \$100$, $U = \$500$. You should also submit the listing of a program that implements this algorithm in your language of choice.
4. Suppose M is an adapted process such that $\boldsymbol{E} M_\sigma = \boldsymbol{E} M_0 = M_0$ for every finite stopping time σ. Must M be a martingale? Prove it, or find a counter example.

Assignment 8 (assigned 2020-10-21, due 2020-10-28).

1. Consider the N period binomial model with $0 < d < 1 + r < u$. A *look-back call option* is a European call option with strike price chosen to be the minimum of the stock price over the entire time period.
   (a) Write down an algorithm/formula to price this option and find the trading strategy required to replicate this option.
   (b) Using your algorithm above and a computer, compute the price of this option at time 0 with $N = 30$, $u = 1.1$, $d = .95$, $r = 2\%$, $S_0 = \$15$. Also compute the number of shares in the replicating portfolio at time 0. You should also submit a snippet of the listing of a program that implements this algorithm in your language of choice.
2. Consider infinitely many i.i.d. coin tosses where the probability of tossing heads is p and probability of tossing tails is $q = 1 - p$. Let $d > 0$ and set $u = 1/d$. We start with $S_0 > 0$ and define $S_{n+1} = uS_n$ if the $(n+1)^{\text{th}}$ coin is heads, and $S_{n+1} = dS_n$ if the $(n+1)^{\text{th}}$ coin is tails. Fix $m_0, n_0 \in \mathbb{N}$ and define $L = d^{m_0} S_0$ and $U = u^{n_0} S_0$. Let $\tau = \min\{n \in \mathbb{N} \mid S_n \notin (L, U)\}$.
   (a) Suppose f is a function such that the process $f(S_{\tau \wedge n})$ is a martingale. Find a finite difference equation (FDE) for f.
   (b) If $p \neq q$, find f so that $f(L) = 0$, $f(U) = 1$ and the process $f(S_{\tau \wedge n})$ is a martingale. [Hint: Guess $f(x) = A + Bx^\alpha$, and find A, B, α.]
   (c) Do the previous part when $p = q$. [Hint: Try something with $\log x$.]
   (d) Find $\boldsymbol{P}(S_\tau = U)$. [Even though τ is not bounded, one can show that optional sampling theorem still applies. Feel free to assume it.]
3. Consider the N period binomial model with $0 < d < 1 + r < u$, and consider an American option with intrinsic value $G = (G_0, G_1, \dots, G_N)$. Define $V_N = G_N$, $V_n = \max\{G_n, \tilde{\boldsymbol{E}}_n V_{n+1}/(1+r)\}$, and $\sigma^* = \min\{n \mid V_n = G_n\}$. Define

$$A_n = \sum_{k=0}^{n-1} \Big(G_k - \frac{1}{1+r} \tilde{\boldsymbol{E}}_k V_{k+1} \Big)^+ (1+r)^{n-k}, \quad \text{and} \quad X_n = V_n + A_n \,.$$

   (a) Show that $\frac{1}{1+r} A_{n+1} = A_n + (G_n - \frac{1}{1+r} \tilde{\boldsymbol{E}}_n V_{n+1})^+$ and $X_n = \frac{1}{1+r} \tilde{\boldsymbol{E}}_n X_{n+1}$.
   (b) Show that $X_n \geqslant G_n$ for all n and $X_{\sigma^*} = G_{\sigma^*}$.
   (c) Show that $X_n > G_n$ if $n < \sigma^*$.
   (d) Conclude (X_n) is the wealth process of the replicating portfolio of an American option with intrinsic value G, and that σ^* is the minimal optimal exercise time. [We will give an alternate proof of this in class, without using the above explicit formula.]

Assignment 9 (assigned 2020-10-28, due Never).

1. Consider the N period binomial model with $0 < d < 1 + r < u$, and let S_n denote the stock price at time n. An American put, call and straddle options with strike K have intrinsic value $G_n^P = (K - S_n)^+$, $G_n^C = (S_n - K)^+$ and $G_n^S = |S_n - K|$ respectively. A European put call and straddle with strike K and maturity N have payoffs G_N^P, G_N^C and G_N^S respectively.
   (a) Write down an algorithm/formula to price these options.
   (b) Can you express the arbitrage free price of the European put in terms of the European call and straddle? Justify / explain.
   (c) Are the American options at least as expensive as the European options? If yes, prove it. If no, find a counter example.
   (d) Choose $S_0 = \$8$, $N = 3$, $u = 2$, $d = 1/2$, $r = 1/4$ and $K = 4$. Is there a difference between the price of the American call and European call? Explain.
   (e) With parameters as in the previous part, is there a difference between the price of the American put and European put? Explain.
   (f) With parameters as in the previous part, is the price of the American straddle the sum of that of the American put and American call? Explain.
   (g) Choose $N = 50$, $u = 1.1$, $d = .9$, $r = 1\%$, $S_0 = \$10$, $N = 50$ and $K = (1+r)^N S_0$. Use a computer to find the prices of the American put and European put at time 0. Also find the number of shares at time 0 held in the replicating portfolio of each of these options. If σ^* is the minimal optimal exercise policy for the American put, find $\max\{S_5(\omega) \mid \omega \in \Omega,\ \sigma^*(\omega) \leqslant 5\}$.
   (h) Do the previous part for $N = 500$. (Note, there is a catch with this: if you modified the code I gave you for the up and rebate options it will almost surely not run in finite time. You need to figure out why, and if you can do it differently.)
2. Consider the N period binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value $G_n = g_n(S_n)$. You are given $g_N(x) = (x - K)^+$, and that *every* exercise policy is optimal. Find finite difference equations for the functions g_n.
3. Consider N i.i.d. coin tosses that land heads with probability p and tails with probability $q = 1 - p$. Let $0 < d < u$, $S_0 > 0$, and define $S_{n+1} = uS_n$ if the $(n+1)^{\text{th}}$ coin is heads, and $S_{n+1} = dS_n$ otherwise.
   (a) Suppose for some sequence of functions f_0, f_1, …, f_n, the process $f_n(S_n)$ is a martingale. Find a recurrence relation between these functions.
   (b) Given a function g, use the previous part to write down a system of recurrence relations that allows you to compute $\boldsymbol{E} g(S_N)$.
   (c) Given $\alpha \in \mathbb{R}$ compute $\boldsymbol{E} S_N^\alpha$.

Assignment 10 (assigned 2020-11-04, due 2020-11-11).

1. Do question 1 from the previous homework.
2. Consider the N period binomial model with $0 < d < 1 + r < u$, and consider an American option with intrinsic value $G = (G_0, \dots, G_N)$. Let V_n be the arbitrage free price of this option at time n. Let $D_n = (1+r)^{-n}$, and $D_n V_n = D_n X_n - A_n$ be the Doob decomposition of the $\tilde{P}$ super martingale DV.
   (a) If for some finite stopping time σ we have $V_\sigma = G_\sigma$, must σ be an optimal exercise time? If yes, prove it. If no, find a counter example.
   (b) If for some finite stopping time σ we have $X_\sigma = G_\sigma$, must σ be an optimal exercise time? If yes, prove it. If no, find a counter example.
3. You are an election forecaster for a race between two candidates A and B. Let V_n^A be the number votes for A after the first n people have voted, and let $V_n^B = n - V_n^A$, $X_n = V_n^A - V_n^B$. In our model, we dictate that the probability that the $(n+1)^{\text{th}}$ person has voted for A is V_n^A / n.
   (a) Let $N \in \mathbb{N}$, $n \in \{1, \dots, N\}$, $x \in \{-n, \dots, n\}$ and define $f_n(x) = \boldsymbol{P}(X_N > 0 \mid X_n = x)$. That is, $f_n(x)$ is the chance that candidate A has more votes after N people have voted given that after n people have voted candidate A has x votes more than candidate B. Find f_N, and a recurrence relation that allows you to compute f_n.
   (b) Suppose now $N = 100,000$, and after $10,000$ people have voted candidate A leads by 100 votes. What is the probability he wins, correct to 4 decimal places? (You may use a computer / spreadsheet.)

Assignment 11 (assigned 2020-11-11, due 2020-11-18).

1. ~~Consider the N period binomial model with $0 < d < 1 + r < u$, $r > 0$, and let S_n denote the stock price. For $K \geqslant 0$, let τ_K^* denote the minimal optimal exercise time for an American put with strike K.~~
   (a) ~~If $K \leqslant K'$, show that $\tau_K^* \geqslant \tau_{K'}^*$ almost surely.~~
   (b) ~~Show that $\lim_{K \to \infty} \tau_K^* = 0$ almost surely.~~
   (c) ~~Show that there exists K_0 such that for all $K \geqslant K_0$ we have $\tau_K^* = 0$ almost surely.~~
2. Let $G = (G_0, \dots, G_N)$ be an adapted process, and let V be the Snell super-martingale envelope of G and $V = M - A$ be the Doob decomposition of V. Recall we say that a finite stopping time τ^* solves the optimal stopping problem for G if $\boldsymbol{E} G_{\tau^*} \geqslant \boldsymbol{E} G_\tau$ for all finite stopping times τ.
   (a) If τ^* solves the optimal stopping problem for G, then must $A_{\tau^*} = 0$ and $V_{\tau^*} = G_{\tau^*}$? Prove it, or find a counter example.
   (b) Conversely, if $A_{\tau^*} = 0$ and $V_{\tau^*} = G_{\tau^*}$, must τ^* solve the optimal stopping problem for G? Prove it, or find a counter example.
   (c) Is the previous subpart true if we only assume $V_{\tau^*} = G_{\tau^*}$? Prove it, or find a counter example.
   (d) Is part (b) true if we only assume $A_{\tau^*} = 0$? Prove it, or find a counter example.
   (e) If σ^* and τ^* are two solutions to the optimal stopping problem for G then must $\sigma^* \wedge \tau^*$ also be a solution to the optimal stopping problem for G?
3. Let $G = (G_0, \dots, G_N)$ be an adapted process, and fix $k \in \{0, \dots, N\}$. Let V be the Snell super-martingale envelope of G, and define $\sigma_k^* = \min\{n \geqslant k \mid V_n = G_n\}$. If σ_k is any stopping time such that $\sigma_k \geqslant k$, then show that $\boldsymbol{E}_k G_{\sigma_k^*} \geqslant \boldsymbol{E}_k G_{\sigma_k}$ almost surely. (I've proved this for $k = 0$ in class when I was doing American options. I stated, but did not prove this for $k > 0$.)

Assignment 12 (assigned 2020-11-18, due 2020-12-02).

1. In the Binomial model we characterized the wealth of self financing portfolios as those for which $X'_{n+1} = \Delta'_n S'_{n+1} + (1+r)(X'_n - \Delta'_n S'_n)$, where S'_n denotes the stock price at time n and Δ'_n is adapted. In the multiple asset model, we characterized trading strategies as those for which $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$, where $\Delta_n = (\Delta^0_n, \dots, \Delta^d_n)$ is adapted, and $S_n = (S^0_n, \dots, S^d_n)$ is the vector of asset prices, with S^0 denoting the risk free asset (bank). Now set $d = 1$, $S^0_n = (1+r)^n$, $S^1_n = S'_n$. Given a portfolio with wealth process X'_n holding Δ'_n shares of the stock (in the Binomial model), write down a formula for the trading strategy Δ_n in the vector notation from the multiple asset model. Use this to show that a portfolio is self-financing (as we defined for the Binomial model) if and only if the corresponding trading strategy is self-financing (as we defined in the multiple asset model).
2. Let $M \in \mathbb{N}$, $\bar{Q} = \{v \in \mathbb{R}^M \mid v_1 \geqslant 0, \dots, v_M \geqslant 0\}$, $\mathring{Q} = \{v \in \mathbb{R}^M \mid v_1 > 0, \dots, v_M > 0\}$. Let $V \subseteq \mathbb{R}^M$ be a subspace with $\dim(V) = M - 1$. Show that V has a unique unit normal vector in $\mathring{Q}$ if and only if $V \cap \bar{Q} = \{0\}$ ~~and $\dim(V) = M - 1$~~.
[I stated, but did not prove this in class. If $\dim(V) < M - 1$, then proving existence of a normal vector in $\mathring{Q}$ is tricky and uses the Hyperplane separation theorem. If $\dim(V) = M$ then the proof is much simper, and is what is asked of you in this question.]
3. Let $\Omega = \{1, \dots, M\}^N$ represent a probability space of N rolls of M-sided dies, and consider the multiple asset model with d stocks and a bank. Let $S = (S^0, \dots, S^d)$ denote the vector process of asset prices with S^0 being the price process of the bank. Fix $\omega' = (\omega_1, \dots, \omega_n)$, and set
$$U = \{(\Delta_n(\omega') \cdot S_{n+1}(\omega', 1), \dots, \Delta_n(\omega') \cdot S_{n+1}(\omega', M)) \mid \Delta_n(\omega') \in \mathbb{R}^{d+1}\},$$
$$V = \{(\Delta_n(\omega') \cdot S_{n+1}(\omega', 1), \dots, \Delta_n(\omega') \cdot S_{n+1}(\omega', M)) \mid \Delta_n(\omega') \cdot S_n(\omega') = 0\}.$$
   (a) Show that U, V are (vector) subspaces of $\mathbb{R}^M$, and $\dim(V) \leqslant d$.
   (b) Suppose the market is arbitrage free. Show that $\dim(V) = M - 1$ if and only if $\dim(U) = M$.
   (c) Must $\dim(V) = \dim(U) - 1$? Prove it, or find a counter example.
4. (a) Let $\mathcal{Q} = \{\tilde{\boldsymbol{P}}^\alpha\}$ be the set of all risk neutral measures. For each $\tilde{\boldsymbol{P}}^\alpha \in \mathcal{Q}$, let $\tilde{p}^\alpha$ be its probability mass function. Given $\theta \in [0,1]$, and $\tilde{\boldsymbol{P}}^\alpha, \tilde{\boldsymbol{P}}^\beta \in \mathcal{Q}$ define the measure $\theta\tilde{\boldsymbol{P}}^\alpha + (1-\theta)\tilde{\boldsymbol{P}}^\beta$ to be the probability measure with probability mass function $\theta\tilde{p}^\alpha + (1-\theta)\tilde{p}^\beta$. For every $\theta \in [0,1]$, and $\tilde{\boldsymbol{P}}^\alpha, \tilde{\boldsymbol{P}}^\beta \in \mathcal{Q}$ must $\theta\tilde{\boldsymbol{P}}^\alpha + (1-\theta)\tilde{\boldsymbol{P}}^\beta \in \mathcal{Q}$? (I.e. is the set of all risk neutral measures a convex set?)
   (b) Consider an arbitrage free market with d risky assets and a bank. Consider a new security on this market, that may or may not be replicable. Let $\mathcal{V}_0$ be the set of all possible arbitrage free prices of this security at time 0. Must $\mathcal{V}_0$ either be empty, or be an interval? Prove it, or find a counter example.

Assignment 13 (assigned 2020-12-02, due 2020-12-11).

1. Let $0 < d < 1 + r < u$, $d < 1 < u$, and consider a market model consisting of a bank with interest rate r, and a stock with price process S_n defined as follows: We start with $S_0 > 0$. At time n, roll a 3 sided die and set $S_{n+1} = uS_n$ if we roll 1, $S_{n+1} = S_n$ if we roll 2 and $S_{n+1} = dS_n$ if we roll 3.
   (a) Is this market complete? Is it arbitrage free? Prove it.

   Suppose we add a second risky asset to this market whose price is given by $V_n = (S_n - S_0)^+$ for all $n \geqslant 1$.
   (b) If N (the total number of periods) is 1, find all $V_0 \in \mathbb{R}$ for which this market is complete and arbitrage free.
   (c) Suppose now $N = 2$ and $ud < 1$, and $V_0 \in \mathbb{R}$ is any one of the values you found in the previous part. Is this market complete? Is it arbitrage free? Prove or disprove your answer.
2. Let $\mu, \sigma \in \mathbb{R}$ and $X \sim \mathcal{N}(\mu, \sigma^2)$.
   (a) Find $a, b > 0$ such that for all $\lambda > 0$ we have $\boldsymbol{P}(|X - \mu| > \lambda) \leqslant a e^{-b\lambda^2}/\lambda$.
   (b) *(Unrelated)* For any $t \in \mathbb{R}$, compute $M_X(t) = \boldsymbol{E}e^{tX}$.
   (c) Find a relation between derivatives of M_X and the moments $\boldsymbol{E}X^n$. Use this to compute $\boldsymbol{E}(X - \mu)^4$.
3. Suppose (X, Y) is a 2 dimensional normally distributed random variable. If $\operatorname{cov}(X, Y) = 0$, must X and Y be independent? Prove or disprove it.
4. Consider a market with a bank (with continuously compounded) interest rate r, and a stock whose price is modelled by a geometric Brownian motion with mean return rate α and volatility σ. In this market you may trade at any instant of time.
   (a) Find the arbitrage free price of a European put option with strike K and maturity T.
   (b) Let $0 < K_1 < K_2$, and S_t denote the stock price at time t. Find the arbitrage free price of a European option that pays $((S_T \wedge K_2) - K_1)^+$ at maturity T.
5. Following our notation from class, let X_n be i.i.d. random variables with $\tilde{\boldsymbol{E}}X_n = 0$ and $\tilde{\boldsymbol{E}}X_n^2 = 1$. Set $W^N_n = \frac{1}{\sqrt{N}} S_n = \frac{1}{\sqrt{N}} \sum_1^n X_k$, and $W_t = \lim_{N\to\infty} W^N_{\lfloor Nt \rfloor}$. Given a random variable Y recall $\boldsymbol{E}_t Y = \lim_{N\to\infty} \tilde{\boldsymbol{E}}_{\lfloor Nt \rfloor} Y$. Given a bounded continuous function $f = f(x, y)$, $0 \leqslant s \leqslant T$, and $n \geqslant Ns$, we know that there exist functions g^N_n such that $g^N_n(W^N_{\lfloor sN \rfloor}, W^N_n) = \tilde{\boldsymbol{E}}_n f(W^N_{\lfloor sN \rfloor}, W^N_{\lfloor nT \rfloor})$. For $t \in [s, T]$, set $u_t(x, y) = \lim g^N_{\lfloor Nt \rfloor}(x, y)$. Express $u_t(x, y)$ as an integral involving f and the Gaussian. Use this to prove that W_s and $W_T - W_s$ are independent, normally distributed with mean 0 and variances s and $T - s$ respectively.
6. Given $\alpha \in \mathbb{R}$ find β so that $e^{\alpha W_t + \beta t}$ is a martingale.

Assignment 14 (assigned 2020-12-10, due Never).

Note HW 13 (on the previous page) is due 2020-12-11

1. Let X_n be a sequence of infinitely many fair coin flips. A bet of B_n dollars at time n pays $B_n X_{n+1}$ dollars at time $n+1$. The bet B_n may be random but has to be adapted (i.e. B_n can only depend on the first n coin flips).
   (a) Let B_n be an adapted process, and consider a gambler that bets B_n dollars at time n. Let M_n denote the cumulative gain/loss of the gambler up to (and including) time n. By convention, we set $M_0 = 0$. Write down a formula for M_n. Is it a martingale?

   A gambler decides to go *double or nothing*. He bets \$1 at time 0. After that, he doubles his bet if he loses and collects his winnings and stops playing if he wins. Let τ be the time he stops playing, B_n his bet at time n and M_n be his cumulative gain/loss up to (and including) time n.
   (b) Is τ a stopping time? Is τ finite almost surely?
   (c) Compute $\boldsymbol{E} M_n$, $\boldsymbol{E} M_\tau$, $\boldsymbol{E} M_\tau^2$
   (d) We've proved Doob's optional sampling theorem when τ is bounded. Does it apply here?
   (e) More generally Doob's optional sampling theorem also applies when there exists $C \in \mathbb{R}$ such that $\boldsymbol{E}\tau < \infty$ and $\boldsymbol{E}_n(\mathbf{1}_{\tau > n}|X_{n+1} - X_n|) \leqslant C$ for all $n \in \mathbb{N}$. Are either of these conditions satisfied here?
2. Let $p \in (0,1)$, and consider infinitely many coin flips that land heads with probability p and tails with probability $1-p$. Let $a, b, c, d \in \mathbb{R}$ be such that $pa + (1-p)c = 0$ and define

$$u_N = 1 + \frac{a}{\sqrt{N}} + \frac{b}{N}\,, \qquad d_N = 1 + \frac{c}{\sqrt{N}} + \frac{d}{N}\,.$$

   Let S^N be a process such that $S_0^N = S_0$, $S_{n+1}^N = u_N S_n^N$ if the $(n+1)$-th coin lands heads, and $S_{n+1}^N = d_N S_n^N$ if the $(n+1)$-th coin lands tails. Let $S_t = \lim_{N \to \infty} S^N_{\lfloor Nt \rfloor}$. Show that $S_t = S_0 e^{(\alpha - \sigma^2/2)t + \sigma W_t}$, for some $\alpha, \sigma \in \mathbb{R}$ and a Brownian motion W. Find α, σ explicitly in terms of p, a, b, c, d.
3. Consider a market with a bank (with continuously compounded) interest rate r, and a stock whose price is modelled by a geometric Brownian motion with mean return rate α and volatility σ. In this market you may trade at any instant of time. Consider a European call option on the stock with strike K and maturity T. Find the number of shares Δ_t held in the replicating portfolio of this option at time t. [Answer: $\Delta_t = \partial_x c(t, S_t)$. Hint: Write it as the limit of binomial models.]
4. Let W be a Brownian motion.
   (a) If $s, t \geqslant 0$, compute $\boldsymbol{E}(W_s W_t)$.
   (b) Show that $W_t^2 - t$ is a martingale.
5. I may add more problems here over the next few days.