21-370 Discrete Time Finance: Final Part 1.

2020-12-17

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not*, however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *You must record yourself (audio, video and screen) and share it with me as instructed by email.*
- *Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *You have 90 minutes. All questions are worth an equal amount of points. Good luck* ◡̈.

1. Consider N independent rolls of a 3-sided die that rolls 1 with probability 1/6, 2 with probability 1/3 and 3 with probability 1/2. Let $S_0 = 1$, and define $S_{n+1} = 2S_n$ if the $(n+1)^{\text{th}}$ die rolls 1, $S_{n+1} = S_n$ if the $(n+1)^{\text{th}}$ die rolls 2, and $S_{n+1} = S_n/2$ if the $(n+1)^{\text{th}}$ die rolls 3. Find $\boldsymbol{E}(S_n^2)$ for $n = 21370$.

2. Consider the N period Binomial model with $N = 4$, $u = 2$, $d = 1/2$, $r = 1/4$. Let σ be the first time two consecutive coins flip tails. A "no-regrets" option pays $S_{\sigma-2}$ at time σ, if $\sigma \leqslant N$, and pays S_N at time N otherwise. What is the arbitrage free price of this option at time 0? (Here $S_{\sigma-2}$ is the stock price at time $\sigma - 2$.)

3. State whether each of the following is true or false. No proof is required. A correct answer is worth full credit, a blank answer is worth half credit, and an incorrect answer is worth no credit.

   (a) If M is an adapted process such that $M_{n+1} - M_n$ has mean 0 and is independent of $\mathcal{F}_n$, then M is a martingale.

   (b) If M is a martingale then $M_{n+1} - M_n$ has mean 0 and is independent of $\mathcal{F}_n$.

   (c) Let $G = (G_0, \dots, G_N)$ be some adapted process. Suppose σ^* and τ^* are two solutions to the optimal stopping problem for G with $\sigma^* \leqslant \tau^*$ almost surely. If η is any finite stopping time for which $\sigma^* \leqslant \eta \leqslant \tau^*$ almost surely, then η must also be a solution to the optimal stopping problem for G.

   (d) If σ^* and τ^* are two optimal exercise times for an American option (in a complete arbitrage free market), then so is $\sigma^* \vee \tau^*$. (Here $\sigma^* \vee \tau^* = \max\{\sigma^*, \tau^*\}$.)

   (e) Consider a market with $d+1$ assets with price processes S^0, S^1, …, S^d. Let X_n denote the wealth of a self financing portfolio at time n. There exists a *predictable* process $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ such that $X_n = \Delta_n \cdot S_n$.

4. Let W be a Brownian motion, $0 < s < t$. Compute $\boldsymbol{E}_s(W_t^3)$. Express your answer in terms of s, t, W_s without involving integrals, expectations or conditional expectations.

21-370 Discrete Time Finance: Final Part 2.

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1. Consider a financial market with a stock and a money market account. The money market has interest rate $r = 10\%$. The stock price at time 0 is $S_0 = \$10$. At time 1, the stock price is determined by the roll of a fair, 3 sided die as follows: If the die rolls $i \in \{1, \dots, 3\}$, then the stock price at time 1 is given by $S_1 = iS_0 = 10i$. Consider a new security that pays $V_1 = \$1$ at time 1 if the die rolls 1, and 0 otherwise. Find all $V_0 \in \mathbb{R}$ such that if we are allowed to additionally trade this new security for price V_0 at time 0, then the market remains arbitrage free.

2. Consider the Binomial model with 3 periods, and parameters $u = 3/2$, $d = 3/4$, $r = 1/4$, and let the stock price at time n is denoted by S_n. At time 0 the stock price is \$8. Consider American put option on this stock with strike price \$8.

   (a) Find the arbitrage free price of this option at time 0. (Express your answer, and all intermediate calculations as fractions.)

   (b) Let σ^* be the minimal optimal exercise time of this option. Find $\tilde{\boldsymbol{P}}(\sigma^* = 3)$.

3. Let $X_1, X_2, \dots,$ be a sequence of i.i.d. random variables such that $\boldsymbol{E}X_k = 0$ and $\boldsymbol{E}X_k^2 = 1$. Let $S_n = \sum_{k=1}^n X_k$. Find $\lim_{n\to\infty} \boldsymbol{P}\Big(\Big|\frac{S_n}{n^{3/4}}\Big| > 2\Big)$. Justify your answer.

4. Consider infinitely many i.i.d. coin tosses where the probability of tossing heads is 1/2 and the probability of tossing tails is 1/2. Let $X_n = 1$ if the n^{th} coin is heads, and $X_n = -1$ otherwise. Let $M \in \mathbb{N}$, and suppose $S_0 \in \{1, 2, \dots, M-1\}$. Given $n \in \mathbb{N}$ define $S_{n+1} = S_n + X_{n+1}$, and $\tau = \min\{n \mid S_n \notin (0, M)\}$. Compute $\boldsymbol{E}\tau$.

   Hint: First compute $\boldsymbol{P}(S_\tau = M)$ and $\boldsymbol{P}(S_\tau = 0)$. Next find $\beta \in \mathbb{R}$ such that the process $Y_n = S_n^2 + \beta n$ is a martingale, and use this to compute $\boldsymbol{E}\tau$. Even though the stopping time here is unbounded, you may apply Doob's optional sampling theorem.