LECTURE NOTES ON DISCRETE TIME FINANCE
FALL 2020

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Contents

Note: The page numbers and links will not be correct in the annotated version.

1. **Preface.**

These are the slides I used while teaching this course in 2020. I projected them (spaced out) in class, and filled in the proofs by writing over them with a tablet. Both the annotated version of these slides with handwritten proofs, and the compactified un-annotated version can be found on the class website. The LaTeXsource of these slides is also available on git.

1. Syllabus Overview

- Class website and full syllabus: http://www.math.cmu.edu/~gautam/sj/teaching/2020-21/370-dtime-finance
- TA's: Lily Chen <huipingc@andrew.cmu.edu>, Jose Olvera <joseluim@andrew.cmu.edu>.
- Homework Due: Every Wednesday, before class (on Gradescope)
- Midterms: Wed Sep 30, 5th week, and Wed Nov 4th, 10th week (self proctored, can be taken any time)
- **Zoom lectures:**
  - ▷ Please enable video. (It helps me pace lectures).
  - ▷ Mute your mic when you're not speaking. Use headphones if possible. Consent to be recorded.
  - ▷ If I get disconnected, check your email for instructions.
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ Bottom 20% homework is dropped from your grade (personal emergencies, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Exams:**
  - ▷ Can be taken at any time on the exam day. Open book. Use of internet allowed.
  - ▷ Collaboration is forbidden. You may not seek or receive assistance from other people. (Can search forums; but may not post.)
  - ▷ Self proctored: Zoom call (invite me). Record yourself, and your screen to the cloud.
  - ▷ Share the recording link; also download a copy and upload it to the designated location immediately after turning in your exam.

- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - Not writing up solutions independently and/or plagiarizing solutions
    - Turning in solutions you do not understand.
    - Seeking, receiving or providing assistance during an exam.
    - Discussing the exam on the exam day (24h). Even if you have finished the exam, others may be taking it.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 30% homework, 20% each midterm, 30% final.

2. Replication, and Arbitrage Free Pricing

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ There doesn't exist a trading strategy with $\boxed{X_0 = 0}$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.
- Now consider a non-traded asset Y (e.g. an option). How do you price it?
- *Arbitrage free price:* V_0 is the arbitrage free price of Y, if given the opportunity to trade Y at price V_0, the market remains arbitrage free.

Market. ← { ① Money Market. ② Stock.

NTA: Call option ⟶ at mat pays $(S_N - K)^+$

stock price, strike, mat time.

- How do you compute the arbitrage free price? Replication:
  - ▷ Say the non-traded asset pays V_N at time N (e.g. call options).
  - ▷ Say you can *replicate* the payoff through a trading strategy $X_0, \dots, X_N = V_N$ (using only traded assets).
  - ▷ Then the arbitrage free price is uniquely determined, and must be X_0.

Question 2.1. *Is the arbitrage free price always unique?*

NTA Pays V_N at time N.

Trade → X_0 → wealth at time 0

X_1 → " " 1

⋮

X_N → " " " N

Trade (tradable assets)

Suppose $X_N = V_N$ → AFP of the NTA is X_0

Say V_0 = AFP of the NTA at time 0

Say $V_0 = X_0 + \varepsilon$ ← Wealth at time 0

Pf: $+V_0$ (buy the NTA)

$-X_0$ (sell (by trading strat))

ε (money market).

Wealth at time N $= \boxed{V_N - X_N} + (1+r)^N \varepsilon$, $\boxed{V_N - X_N} = 0$

$\implies \varepsilon = 0$!

Theorem 2.2. *The arbitrage free price is* unique *if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.*

Proof. We already proved that if a replicating strategy exists then the arbitrage free price is unique. The other direction is harder, and will be done later. ☐

Question 2.3. *If a replicating strategy exists, must it be unique?*

NO.

But initial wealth of the replicating strat is unique

(AFP!!)

Last time:

① Arbitrage $\longrightarrow$ X_0 (initial wealth) $= 0$

No arbitrage means $\underbrace{X_n}_{\text{wealth at time } n.} \geq 0$ then $\boxed{X_n = 0}$

② AFP: Traded assets (& M.M.)

Security (NTA) $\longrightarrow$ AFP $= V_0$ if given the option to trade the NTA at price V_0, the market remains arb free.

③ Pricing = Replication!

Question 3.4. *Consider a financial market with a money market account with interest rate r, and a stock. Let $K > 0$. A* forward contract *requires the holder to buy the stock at price K at maturity time N. What is the arbitrage free price at time 0?*

Forward Contract: Req to buy at price K at time N.

① Money Mkt (r)
② Stock.

Q: Compute AFP (time 0).

Q: Payoff at maturity? $\longrightarrow S_N - K$

price at time N

$S_N - K$

Q: Replicate $\longrightarrow$ ① buy 1 share of stock
② short $\frac{K}{(1+r)^N}$ cash

time 0

① 1 stock
② K cash (short).

time N

$$AFP = S_0 - \frac{K}{(1+R)^N}$$

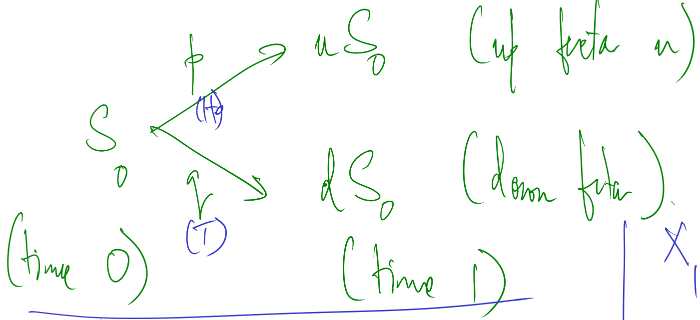
4. Binomial model (one period)

270

Say we have access to a money market account with interest rate r. The *binomial model* dictates that the stock price varies as follows. Let $p \in (0,1)$, $q = 1 - p$, $0 < d < u$ (up and down factors). Flip a coin that lands heads with probability p, and tails with probability q. When the coin lands heads, the stock price changes by the factor u, and when it lands tails it changes by the factor p.

Question 4.1. *When is there arbitrage in this market?*

No arb $\iff d < 1+r < u$



X_0 = Initial wealth.

$\to \Delta_0$ shares of stock

$\to X_0 - \Delta_0 S_0 \to$ cash.

$$X_1 = \text{wealth at time } 1$$
$$= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$$
$$= (1+r) X_0 + (1+r)\Delta_0 \left(\frac{S_1}{1+r} - S_0 \right)$$

No arb: $X_0 = 0$

$$X_1 = \Delta_0 \left(S_1 - (1+r) S_0 \right)$$

Question 4.2. *If a security pays V_1 at time 1, what is the arbitrage free price at time 0. (V_1 can depend on whether the coin flip is heads or tails).*

Get $V_1(H)$ for heads & $V_1(T)$ for tails.

Wealth if heads $= X_1(H) = (1+r)X_0 + (1+r)\Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right)$ ① Want $= V_1(H)$

" " tails $= X_1(T) = (1+r)X_0 + (1+r)\Delta_0\left(\frac{S_1(T)}{1+r} - S_0\right)$ ② Want $= V_1(T)$

Solve → Find $\tilde{p}, \tilde{q}$ $\ni$ $\tilde{p}+\tilde{q}=1$ & $\frac{\tilde{p}S_1(H)+\tilde{q}S_1(T)}{1+r} = \cancel{(1+r)}S_0$

① $\tilde{p}$ + ② $\tilde{q}$ $\Rightarrow$ $\boxed{\frac{\tilde{p}V_1(H)+\tilde{q}(V_1(T))}{1+r} = X_0} + 0$

$$\boxed{X_0 = \tilde{E}\,\frac{V_1}{(1+r)}}$$ ← RNP

AFP.

Δ_0: ① − ② $\Rightarrow$ $\frac{V_1(H) - V_1(T)}{\cancel{1+r}} = \Delta_0\left(\frac{S_1(H)}{\cancel{1+r}} - \frac{S_1(T)}{\cancel{1+r}}\right)$

Find $\tilde{p}, \tilde{q}$: $\tilde{p}\, S_1(H) + \tilde{q}\, S_1(T) = (1+r)\, S_0$

$$\iff \tilde{p}\, u \cancel{S_0} + (1-\tilde{p})\, d \cancel{S_0} = (1+r) \cancel{S_0}$$

$$\tilde{p}(u-d) + d = 1+r \implies \boxed{\tilde{p} = \frac{1+r-d}{u-d}}$$

Note $d < 1+r < u \iff \tilde{p} \in (0,1)$ (can be int as a prob).

Question 4.3. *What's an N period version of this model? Do we have the same formulae?*

$$S_0 \xrightarrow{H} uS_0 \qquad S_0 \xrightarrow{T} dS_0$$

$$uS_0 \to S_2(H,H), \quad uS_0 \to S_2(H,T)$$

$$dS_0 \xrightarrow{H} S_2(T,H), \quad dS_0 \xrightarrow{T} S_2(T,T)$$

No arb?

Guess: $d < (1+r) < u$

Q2!

Unknowns? $\longrightarrow$ 2 $\quad X_0, \Delta_0$

Δ_1 (2 unks)

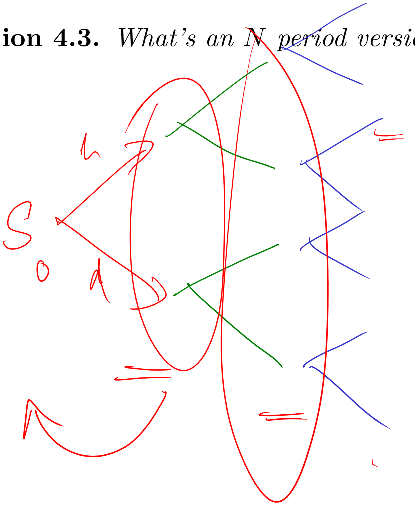
$\delta_2 \to 4$

2^N

outcomes

2^N eq...

Question 4.3. *What's an N period version of this model? Do we have the same formulae?*



Q1 No arb? $(d < 1+r < u)$

Q2: Price Sec

Q3: American options

Can exercise at any time

5. Probability spaces, in our context

Notation $\omega = (\omega_1, \omega_2, \cdots \omega_N) \in \Omega$

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 5.1. The *Sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a } \cancel{\text{coin}} \text{ toss.}\}$ dice roll

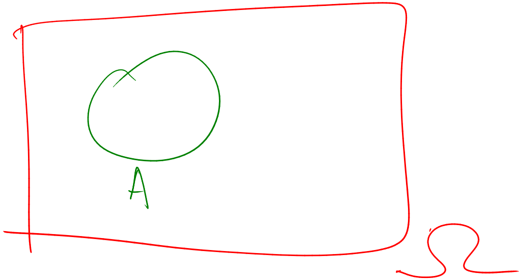
- ▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$.
- ▷ Coins don't have to be identical: Pick $M_1, M_2, \dots, \in \mathbb{N}$, and can require $\omega_i \in \{1, \dots, M_i\}$.

Definition 5.2. A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega$.

Definition 5.3. A *probability mass function* is a function $p \colon \Omega \to [0, 1]$ such that $\boxed{\sum_{\omega \in \Omega} p(\omega) = 1.}$

Definition 5.4. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

$p(\omega)$ = Prob of "ω" occurring.



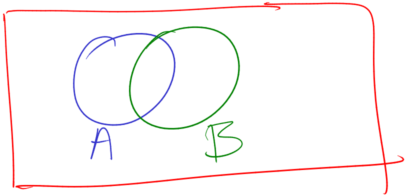
$A \subset \Omega$ (event)

$P(A) = \sum_{\boxed{\omega \in A}} p(\omega)$

5.1. Independence.

Definition 5.5. Two events are independent if $\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)$. (multiplication law)

Question 5.6. *What does it mean for the events $A_1, \ldots, A_n$ to be independent?*



Say A occurs.

Q: Prob B occurs given A occurred

↳ $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Find: $P(A) = P(A|B)$

$$P(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} \iff P(A \cap B) = P(A)P(B)$$

Def : A_1, A_2, A_3 are ind if $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

AND $P(A_1 \cap A_2) = P(A_1)P(A_2)$ & $P(A_2 \cap A_3) = P(A_2)P(A_3)$

$P(A_1 \cap A_3) = P(A_1)P(A_3)$

Def : $A_1, \dots A_N$ are ind if for any sub collection $\{A_{i_1}, A_{i_2}, \dots A_{i_m}\}$,

we have $P\left(\bigcap_{j=1}^{m} A_{i_j}\right) = \prod_{j=1}^{m} P(A_{i_j})$

Definition 5.7. A *random variable* is a function $X : \Omega \to \mathbb{R}$.

(Later will allow $X : \Omega \to \mathbb{R}^d$).

Question 5.8. *What is the random variable corresponding to the outcome of the n^{th} coin toss?*

$\Omega = \{1, 2, \cdots 6\}^N$ $(N \in \mathbb{N})$ $\omega = (\omega_1, \cdots \omega_N)$ & each $\omega_i \in \{1, \cdots 6\}$.

X_2 = RV corresponding to roll of the second die.

Eg: $\boxed{X_2(\omega) = \omega_2}$

(here $\omega = (\omega_1, \omega_2, \cdots \omega_N)$)

ω

ω

Ω

Definition 5.9. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum x_i \boldsymbol{P}(X = x_i)$.

Definition 5.10. The *variance* is $\boldsymbol{E}(X - \boldsymbol{E}X)^2 = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

X is a R.V. $(X : \Omega \longrightarrow \mathbb{R})$

EX = average value of X $= \sum_{\omega \in \Omega} X(\omega)\, p(\omega) = \sum_{i=1}^{M} x_i P(X = x_i)$

Ω finite $\Rightarrow X(\Omega)$ = range of X is finite $= \{x_1, x_2, \dots x_M\}$.

$\{\omega \in \Omega \mid X(\omega) = x_i\} \subseteq \Omega$ (some event)

Notation: $\{X = x_i\} \overset{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x_i\} = X^{-1}(x_i)$

Definition 5.11. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Question 5.12. *What does it mean for the random variables $X_1, \ldots, X_n$ to be independent?*

Question 5.13. *Are uncorrelated random variables independent?*

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

$$\{Y = y\} = \{\omega \in \Omega \mid Y(\omega) = y\}$$

are ind

not enough

$$\boxed{P(X_1 = x_1 \;\&\; X_2 = x_2 \;\cdots\; X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i) \quad \forall x_i}$$

Def: $\forall x_i$, the events $\{X_1 = x_1\}, \{X_2 = x_2\} \ldots \{X_n = x_n\}$ are all ind

(Then $P(X_1 = x_1 \;\&\; X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2)$ etc)

Market $n-1$ assets S^1 --- S^{n-1} & money market

$\boxed{\text{Harrison}}$

$\to \Omega = \{\omega_1, \dots \omega_m\}$

$$S^1(\omega_i) = \_\_\_\_$$

Conditions for ① When is this market complete?

② " " " " arb free?

Last time: A_1 A_2 are ind if $P(A_1 \cap A_2) = P(A_1)P(A_2)$

$\hookrightarrow$ $A_1 \dots A_n$ is ind if $P(A_{i_1} \cap \dots A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k})$

$\forall$ sub collections $\{A_{i_1}, \dots A_{i_k}\} \subseteq \{A_1, \dots A_n\}$.

$\{X = x\}$ ~~=~~ short hand for $\{\omega \in \Omega \mid X(\omega) = x\}$

$$\underline{P(X = x)} = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

Definition 5.11. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Question 5.12. *What does it mean for the random variables $X_1, \ldots, X_n$ to be independent?*

Question 5.13. *Are uncorrelated random variables independent?*

$$\forall x_1 \cdots x_n \in \mathbb{R} \quad P(X_1 = x_1, X_2 = x_2 \cdots X_n = x_n) = P(X_1 = x_1) P(X_2 = x_2) \cdots P(X_n = x_n)$$

looks like not enough.

need $P(X_{i_1} = x_{i_1}, \cdots X_{i_k} = x_{i_k}) = P(X_{i_1} = x_{i_1}) \cdots P(X_{i_k} = x_{i_k})$

$\forall$ sub coll

Turns out is enough

Theorem 5.14. *The random variables $X_1, \ldots, X_n$ are independent if and only if for all $x_1, \ldots, x_n \in \mathbb{R}$ we have*

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2)\cdots\boldsymbol{P}(X_n = x_n)\,. \quad (\forall x_1 \ldots x_n \in \mathbb{R})$$

Corollary 5.15. *Suppose for simplicity all coin tosses have M outcomes (i.e. $\omega_i \in \{1, \ldots, M\}$). Let p be a probability mass function. The coin tosses are all independent, if and only if, there exists functions $p_1, \ldots, p_N$ such that $p(\omega) = p_1(\omega_1)p_2(\omega_2)\cdots p_N(\omega_N)$.*

Check when $n = 3$. Say X, Y, Z are ind. $\Rightarrow \forall x, y, z$

$$P(X = x, Y = y, Z = z) = P(X = x)P(Y = y)P(Z = x).$$

$\Rightarrow$ forward direction.

Converse: Say $\forall x, y, z$, $P(X = x, Y = y, Z = z) = P(X = x)P(Y = y)P(Z = z)$.

Then $\forall x, y \in \mathbb{R}$, $P(X = x, Y = y) = P(X = x)P(Y = y)$

(& similarly for X, Z or Y, Z).

Pf: Say Z takes on the values $z_1, \ldots z_n$.

$$P(X=x, Y=y) = \sum_{i=1}^{n} P(X=x, Y=y, Z=z_i)$$

$$\underline{\text{assumption}} \quad \sum_{i=1}^{n} P(X=x)P(Y=y)\,P(Z=z_i)$$

$$= P(X=x)P(Y=y) \underbrace{\sum_{i=1}^{n} P(Z=z_i)}_{1}$$

QED

General case: Induction. (see std book)

5.2. **Filtrations and adapted processes.**

(Binomial model: $d_i = 2 \ \forall i$)

- Let $N \in \mathbb{N}$, $d_1, \ldots, d_N \in \mathbb{N}$, $\Omega = \{1, \ldots, d_1\} \times \{1, \ldots, d_2\} \times \cdots \times \{1, \ldots, d_N\}$.
- That is $\Omega = \{\omega \mid \omega = (\omega_1, \ldots, \omega_N), \ \omega_i \in \{1, \ldots, d_i\}\}$.
- $d_n = 2$ for all n corresponds to flipping a two sided coin at every time step.

Definition 5.16. We define a *filtration* on Ω as follows: (information)

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

▷ $\mathcal{F}_1 =$ all events that can be described by only the first coin toss (die roll). E.g. $A = \{\omega \mid \omega_1 = H\} \in \mathcal{F}_1$.

▷ $\mathcal{F}_n =$ all events that can be described by only the first n coin tosses. (die rolls)

Question 5.17. *Let* $\Omega = \{H, T\}^3 \cong \{1, 2\}^3$. *What are* $\mathcal{F}_0, \ldots, \mathcal{F}_3$*?*

$\mathcal{F}_0$ = info before rolling any die.

$A_1 = \{\omega \in \Omega \mid \omega = (\omega_1, \ldots \omega_N) \ \& \ \omega_1 = 1\} \in \mathcal{F}_1$

$A_2 = \{\omega \in \Omega \mid \omega = (\quad) \ \& \ \omega_2 = 1\} \notin \mathcal{F}_1$

Claim: # events in $\mathcal{F}_1 = 2^{d_1}$

$A \subseteq \{1, \dots d_1\}$ $\quad (2^{d_1}$ choices for $A)^2$

$$\underbrace{A \times \{1, \dots d_2\} \times \{1, \dots d_3\} \times \dots \{1, \dots d_n\}}\in \mathcal{F}_1$$

$$\mathcal{F}_2 = \Big\{ \underbrace{A}_{2^{d_1}} \times \underbrace{B}_{2^{d_2}} \times \{1, \dots d_3\} \times \dots \{1, \dots d_n\} \Big\}$$

$N = 3$, $\Omega = \underline{\{\pm 1\}^3} = \{\omega \mid \omega = (\omega_1, \omega_2, \omega_3)\ \&\ \omega_i \in \{\pm 1\}\}$.

Enumerate $\underline{\mathcal{F}_0}$, $\mathcal{F}_1$, $\mathcal{F}_2$ & $\mathcal{F}_3$

$\mathcal{F}_0 = \{\phi, \Omega\}$.

$\mathcal{F}_1 = \{\phi, \Omega, \{(1,1,1), (1,-1,1), (1,1,-1), (1,-1,-1)\}$

$, \{(-1,1,1), (-1\ \ \ 1), (-1\ 1), (-1\ 1)\}$.

$\mathcal{F}_2 = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \{(1,1,1), (1,1,-1)\} \cup \ldots$

Note $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$

power set.

Definition 5.18. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \ldots, \omega_n$.

▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.

Question 5.19. *Let* $\boxed{X(\omega) \stackrel{\text{def}}{=} \omega_1 - 10\omega_2}$. *For what n is $\mathcal{F}_n$-measurable?*

$\{X \in [0,1]\}$

$X(\omega) = \omega_1 - 10\omega_2$

Is X $\mathcal{F}_0$ - measurable? NO

" " $\mathcal{F}_1$ - " ? NO

" " $\mathcal{F}_2$ - " ? YES

" " $\mathcal{F}_3$ " ? YES.

$$\Omega = \{-1, 0, 1\}^2$$

$\{(1,0), (1,1), (1,-1)\}$

$\mathcal{F}_1 = 8$ elements

$\overset{1}{\phi} \in \mathcal{F}_1, \quad \overset{2}{\Omega} \in \mathcal{F}_1, \quad \overset{3}{\{(0,0), (0,1), (0,-1)\}} \in \mathcal{F}_1$

4 $\left\{ \begin{matrix} (0,0), & (0,1), & (0,-1) \\ (1,0), & (1,1), & (1,-1) \end{matrix} \right\} \in \mathcal{F}_1$

$\boxed{1^{st} \text{ roll} \longrightarrow 0 \text{ or } 1}$

2^{nd} roll $\longrightarrow$ anything

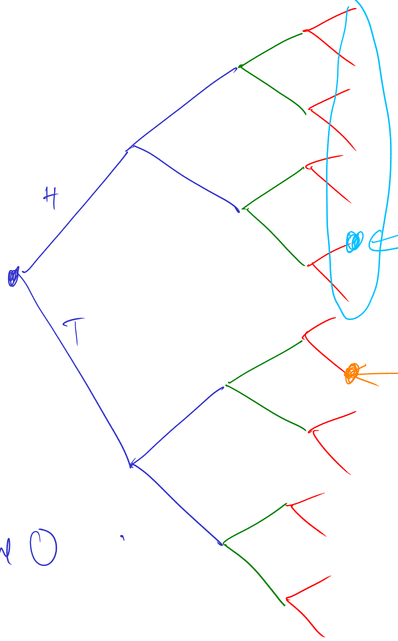
Last time:

$\Omega \to$ Prob space N die rolls.

$\omega = (\omega_1, \cdots \omega_N)$

Filtration $\mathcal{F}_n$ = all events that can be determined using only the first $\boxed{n}$ coin tosses

X is $\mathcal{F}_n$-meas if $X(\omega)$ only depends on $\omega_1, \omega_2 \cdots \omega_n$ (and not $\omega_{n+1} \cdots \omega_N$).



$\in A$

say $\omega \in A$

$\omega = (H, T, T, H)$

(T, H, H, T)

time 0

$A \in \mathcal{F}_1 \to A \subseteq \Omega$

A only depends on the 1^{st} coin toss.

$A = \{(H, T, T, H)\}$

$\Omega = \{$end pts of all these trees$\}$

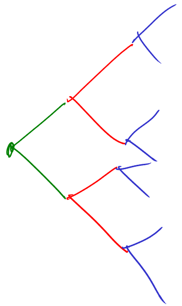
5.3. Conditional expectation.

Definition 5.20. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \frac{\sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')}{\sum_{\omega' \Pi_n(\omega)} p(\omega')}, \qquad \text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \; \dots, \omega'_n = \omega_n\}$$

Remark 5.21. $\boldsymbol{E}_n X$ is the “best approximation” of X given only the first n coin tosses.

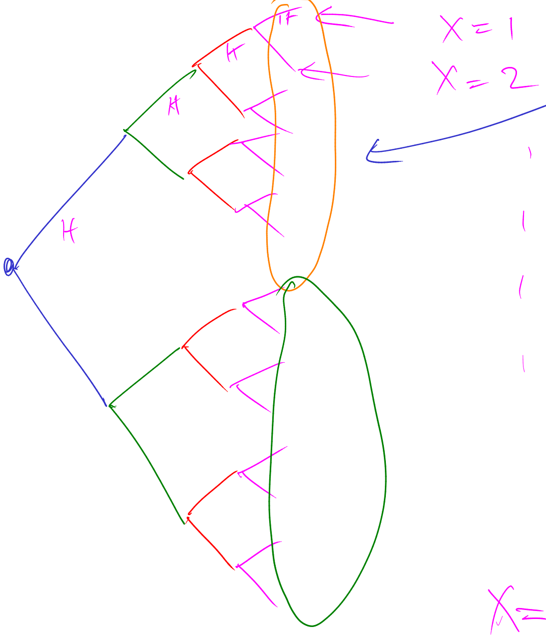
Remark 5.22. The above formula does not generalize well to infinite probability spaces. We will develop a definition that does generalize; after we have that definition we will never ever ever use this formula.



$\omega \in \Omega$. $\Pi_1(\omega) = \{ \text{all } \omega' \in \Omega \mid \omega' = (\omega'_1, \omega'_2, \omega'_3)$

$\& \; \omega'_1 = \omega_1 \}$

$\omega \to (\omega_1, \omega_2, \omega_3)$



$\Pi_1((H, -))$ $\quad X(\omega)$ (fair coin)

$E_1 X$ = cond exp of X given $\mathcal{F}_1$

= best approx of X given only the first coin toss

① If 1st coin is Head

$$E_1 X(\omega) = \frac{\displaystyle\sum_{\omega \in \text{top orange block}} X(\omega) \cdot p(\omega)}{\displaystyle\sum_{\omega \in \text{top orange block}} p(\omega)}$$

② If 1st coin is tails

$X = 16$

$E_1 X(\omega)$ = same form but sum over the bottom green block instead.

Remark 5.23. The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:

(1) $\boldsymbol{E}_n X$ is an $\mathcal{F}_n$-measurable *random variable.* ⇐ $E_n X(\omega)$ only depends on $\omega_1, \omega_2 \cdots \omega_n$ (& not $\omega_{n+1}, \cdots \omega_N$).

(2) For every $A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

$A \in \mathcal{F}_n$ → A some event that can be described in terms of the first n coin tosses done.

Q: Avg of X on the event A = $\displaystyle\sum_{\omega \in A} X(\omega)\, p(\omega)$ / $\displaystyle\sum_{\omega \in A} p(\omega)$

Q: Avg of $E_n X$ on the event A $\displaystyle = \sum_{\omega \in A} E_n X(\omega)\, p(\omega)$ / $\displaystyle\sum_{\omega \in A} p(\omega)$

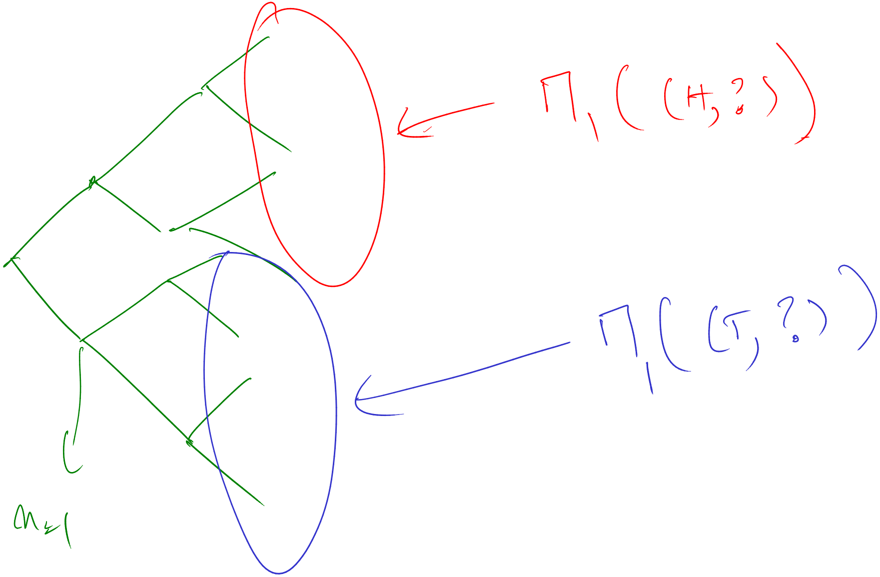
(: Venn's $A \in \mathcal{F}_n$)

① Check $E_n X$ (by my formula) satisfies ① & ② above.

② Use this as my def of cond Exp

Proof of (2):

(1) If $A \in \mathcal{F}_n$, then there exist $\omega^1, \dots, \omega^k \in \Omega$ such that A is the *disjoint union* of $\boxed{\Pi_n(\omega^1), \dots, \Pi_n(\omega^k)}$.



(2) For any $\omega \in \Omega$, $\displaystyle\sum_{\omega' \in \Pi_n(\omega)} \boldsymbol{E}_n X(\omega') p(\omega') = \sum_{\omega' \in \Pi_n(\omega)} X(\omega') p(\omega')$

$$\sum_{\omega' \in \Pi_n(\omega)} \underbrace{\left(\frac{\displaystyle\sum_{\omega'' \in \Pi_n(\cancel{\omega'})} X(\omega'') p(\omega'')}{\displaystyle\sum_{\omega'' \in \Pi_n(\cancel{\omega'})} p(\omega'')} \right)}_{\text{ind of } \omega'} p(\omega') =$$

QED

(3) Hence $\sum_{\omega\in A} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{i=1}^{k} \sum_{\omega\in\Pi_n(\omega^i)} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{i=1}^{k} \sum_{\omega\in\Pi_n(\omega^i)} X(\omega)p(\omega) = \sum_{\omega\in A} X(\omega)p(\omega)\,.$

Definition 5.24. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of X given* $\boxed{\mathcal{F}_n}$, denoted by $\boldsymbol{E}_n X$, or $\boxed{\boldsymbol{E}(X \mid \mathcal{F}_n)}$, to be the unique *random variable* such that:

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.
(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

Remark 5.25. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

Remark 5.26 (Uniqueness). If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega \in A} Y(\omega)p(\omega) = \sum_{\omega \in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.

Definition 5.24. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of X given $\mathcal{F}_n$*, denoted by $\boldsymbol{E}_n X$, or $\boldsymbol{E}(X \mid \mathcal{F}_n)$, to be the unique *random variable* such that:

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.

(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

$\boxed{E_n X = \text{best approx of } X \text{ as a } \mathcal{F}_n \text{ meas R.V.}}$

Remark 5.25. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

Risk Neutral pricing formula

Given $V_N \to$ payoff at maturity of some security

AFP at time $n \leq N$? RNP formula

Cond exp under risk Neutral meas. $\longrightarrow$ $V_n = \tilde{E}_n\big(V_N\ (\text{discount factor})\big)$

Prop: ~~Remark~~

Remark 5.26 (Uniqueness). If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega\in A} Y(\omega)p(\omega) = \sum_{\omega\in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.

Pf: Let $X = Y - Z$. $\Rightarrow$ X is $\mathcal{F}_n$-meas & $\forall A \in \mathcal{F}_n$, $\sum_{\omega\in A} X(\omega)p(\omega) = 0$

Let $A = \{X > 0\}$ (Note $A \in \mathcal{F}_n$ $\because$ X is $\mathcal{F}_n$ meas).

Then by assumption $\sum_{\omega\in A} X(\omega)p(\omega) = 0$

Also, $\underbrace{\sum_{\omega\in A} X(\omega)p(\omega)}_{=0} \geq \underbrace{\left(\min_{\omega\in A} X(\omega)\right)}_{>0} \overbrace{\sum_{\omega\in A} p(\omega)}^{P(A).}$ $\leftarrow$ only possible if $P(A) = 0$

|||^{ly} $B = \{X < 0\} \longrightarrow$ get $P(B) = 0$. $\therefore P(X \neq 0) = 0 \Leftrightarrow P(Y = Z) = 1$

QED.

Theorem 5.27. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$. (On homework).*
(2) If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$. (Tower Property)

Proof: Claim ① $E_m E_n X$ is $\mathcal{F}_m$-meas

② $\forall A \in \mathcal{F}_m$, $\sum_{\omega \in A} E_m E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$ $\Rightarrow E_m E_n X = E_m X$ $\Rightarrow$ QED

Pf of ① $\longrightarrow$ True because $E_m(\;\;)$ is $\mathcal{F}_m$-meas.

② $\rightarrow \sum_{\omega \in A} E_m(E_n X(\omega)) p(\omega) = \sum_{\omega \in A} E_n X(\omega) p(\omega)$ ($\because A \in \mathcal{F}_m$)

$= \sum_{\omega \in A} X(\omega) p(\omega)$ ($\because A \in \mathcal{F}_n$)

QED.

(3) If $\boxed{X \text{ is } \mathcal{F}_n \text{ measurable,}}$ and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_nY$.

Pf: Will show ① $X E_n Y$ is an $\mathcal{F}_n$ meas RV

② $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} X(\omega) E_n Y(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) Y(\omega) p(\omega)$

Pf of ① $\to$ True bc X is $\mathcal{F}_n$ meas & $E_n Y$ is $\mathcal{F}_n$-meas

(X takes on vals $x_1 \dots x_k$).

② Pf: $\sum_{\omega \in A} X(\omega) E_n Y(\omega) p(\omega) = \sum_{i=1}^{k} \sum_{\omega \mid X(\omega) = x_i \cap A} X(\omega) E_n Y(\omega) p(\omega)$

$= \sum_{i=1}^{k} \sum_{\omega \in \{X = x_i\} \cap A} x_i E_n Y(\omega) p(\omega).$

$$= \sum_{i=1}^{k} x_i \left(\sum_{\omega \in \{X = x_i\} \cap A} E_n Y(\omega)\, p(\omega) \right)$$

X is $\mathcal{F}_n$-meas.

$$\Downarrow$$

$$\left(\therefore \underline{\{X = x_i\} \in \mathcal{F}_n} \right)$$

$$= \sum_{i=1}^{k} x_i \left(\sum_{\omega \in \{X = x_i\} \cap A} Y(\omega)\, p(\omega) \right)$$

$$= \sum_{i=1}^{k} \sum_{\omega \in \{X = x_i\} \cap A} x_i Y(\omega)\, p(\omega)$$

$$= \sum_{i=1}^{k} \sum_{\omega \in \{X = x_i\} \cap A} X(\omega) Y(\omega)\, p(\omega) = \sum_{\omega \in A} X(\omega) Y(\omega)\, p(\omega)$$

QED.

Theorem 5.28. *If X is independent of $\mathcal{F}_n$ then* $\boxed{\boldsymbol{E}_n X = \boldsymbol{E} X}$.

$E_n X$ $\to$ $E(X \mid \mathcal{F}_n)$

$P(X \mid A)$ ent

Q6 on HW

$$E(X-Y)^2 \geq E(X - E_n X)^2 \quad \forall\ \mathcal{F}_n\text{-meas } Y.$$

Hint $E(X-Y)^2 = E\left(\underbrace{(X - E_n X)} + \underbrace{(E_n X - Y)}\right)^2$ & expand.

Last time: Cond Exp prop: $\boxed{E_n X = E(X \mid \mathcal{F}_n)}$

(Def) ① $E_n X$ → ⓐ $\mathcal{F}_n$ - meas RV & ⓑ $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$

→ ② $E_n(X + \alpha Y) = E_n(X) + \alpha E_n(Y)$ (X, Y RV's & $\alpha \in \mathbb{R}$)

(on HW)

→ ③ (Tower) $m \leq n \Rightarrow E_m(E_n X) = E_m(X)$

④ If X is $\mathcal{F}_n$ meas & Y is anything then $E_n(XY) = X(E_n Y)$

Theorem 5.28. *If X is $\boxed{\text{independent}}$ of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$* (almost surely)

(If X is $\mathcal{F}_n$-meas, then $E_n X = X$)

X is indep of the first n die rolls

① Def: We say X is ind of $\mathcal{F}_n$ if $\forall A \in \mathcal{F}_n$ & $B \subseteq \mathbb{R}$ the events A & $\{X \in B\}$ are ind.

$\underbrace{A \in \mathcal{F}_n}_{\text{events det from } 1^{st}\ n \text{ die rolls}}$

$B \subseteq \mathbb{R}$

$\{X \in B\} \leftarrow$ some event

$\underbrace{\{\omega \mid X(\omega) \in B\}}_{\text{evts det by } X}$

Remk: Ω finite $\Rightarrow$ ① $\iff$ $\forall a \in \mathbb{R}$ $\forall A \in \mathcal{F}_n$ the events A & $\{X = x\}$ are ind

Thm: X ind of $\mathcal{F}_n \Rightarrow \underbrace{E_n X}_{\mathcal{F}_n\text{-meas RV}} = \underbrace{EX}_{\text{number}}$ (almost surely)

i.e. $E_n X$ is the const EX almost surely!

$\boxed{\text{Conv is false! } E_n X = EX \not\Rightarrow X \text{ is ind of } \mathcal{F}_n}$

Pf: ① Check EX is an $\mathcal{F}_n$-meas RV (true! indep of all dice rolls).

② Check $\forall A \in \mathcal{F}_n$,

$$\sum_{\omega \in A} \underbrace{E_{\cancel{n}} X(\omega)}_{EX \text{ (no } n)} p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$$

Pf: Say X takes on values $x_1, \dots, x_m$

Then $\displaystyle\sum_{\omega \in A} X(\omega)\,p(\omega) = \sum_{k=1}^{m} x_k \underbrace{P(\{X = x_k\} \cap A)}$

$$= \sum_{k=1}^{m} x_k P(X = x_k)\,P(A) \qquad (\because X \text{ is ind of } \mathcal{F}_n)$$

$$= P(A) \underbrace{\sum_{k=1}^{m} x_k P(X = x_k)}_{EX} = P(A)(EX)$$

$$= \sum_{\omega \in A} p(\omega)\, EX \qquad \text{QED !!}$$

Indep lemma:

1. X is $\mathcal{F}_n$ meas $\Rightarrow$ $\overline{E_n X = X}$ (a.s)
2. Y indep of $\mathcal{F}_n$ $\Rightarrow$ $E_n Y = EY$ (a.s.)
3. Q: $E_n f(X, Y)$ for some fn f (e.g. $E_n (X+Y)^{10}$)

"leave alone" "average Y"

Claim: $E_n f(X, Y) = \overbrace{\sum_k f(X, y_k) \, P(Y = y_k)}$

Theorem 5.29 (Independence lemma). *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f \colon \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X,Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

5.4. Martingales.

← think of X_n = RV at time n

Definition 5.30. A *stochastic process* is a collection of random variables $X_0, X_1, \ldots, X_N$.

Definition 5.31. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Question 5.32. *Is* $X_n(\omega) = \sum_{i \leqslant n} \omega_i$ *adapted?* YES ($\omega_i \in \pm 1$ (i^{th} coin toss))

Question 5.33. *Is* $X_n(\omega) = \omega_n$ *adapted? Is* $\boxed{X_n(\omega) = 15}$ *adapted?* *Is* $X_n(\omega) = \omega_{15}$ *adapted? Is* $X_n(\omega) = \omega_{N-i}$ *adapted?*

Remark 5.34. We will always model the price of assets by *adapted* processes. We will also only consider trading strategies which are adapted.

YES. YES $\omega = (\omega_1, \omega_2, \ldots \omega_N)$

& X_0 is $\mathcal{F}_0$-meas, $\mathcal{F}_0 = \{\phi, \Omega\}$

① A RV is $\mathcal{F}_0$ meas $\iff$ it is const.

② Constants are $\mathcal{F}_n$-meas $\forall n \geq 0$.

③ X_n adapted $\Rightarrow$ X_n is $\mathcal{F}_n$ meas $\forall n$. Since $\mathcal{F}_n \subseteq \mathcal{F}_m$ $\forall m \geq n \Rightarrow$

X_n is $\mathcal{F}_m$-meas $\forall\, m \geq n$

Q: $X_n = \omega_{15}$ $\forall n$. Is X adapted? NO

Example 5.35 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.) (Adapted)

Example 5.36. Suppose $\Omega = \{\pm 1\}^N \cong \{H, T\}^N \cong \{1, 2\}^N$. Let $S_0 = a \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

Is S_n adapted? (Used to model stock price in the multi-period Binomial model.)

$\omega = (\omega_1, \omega_2, \ldots, \omega_N)$

Yes.

Definition 5.37. $\boxed{\text{We say an adapted process } X_n}$ is a martingale if $\underbrace{\boldsymbol{E}_n X_{n+1} = X_n}$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.38. Intuition: A martingale is a "fair game".

Question 5.39. *If $m \leqslant n$, is $\boldsymbol{E}_m X_n = X_m$?* ← YES

$$E_n X_{n+1} = X_n$$

Best approx of X_{n+1} given only the first n-die rolls is X_n itself!

Q: $m \leq n$: $E_m X_n = X_m$?

$$E_n X_{n+2} \underset{\text{Tower}}{=} E_n \underbrace{E_{n+1} X_{n+2}}_{\text{mg}} \overset{\text{mg}}{=} E_n (X_{n+1}) \overset{\text{mg}}{=} X_n .$$

QED

Question 5.40. *If M is a martingale does $\boldsymbol{E}M_n$ change with n?*

Know $E_n M_{n+1} = M_n$ $\left(\Leftrightarrow \forall m \leq n,\ E_m M_n = M_m \right)$

Claim: $E M_{n+1} = E M_n$ $\quad$ (E not E_n).

Pf: $E M_{n+1} \underset{\substack{\uparrow \\ \text{def of } E_n}}{=} E E_n M_{n+1} \underset{\substack{\uparrow \\ \text{mg}}}{=} E M_n$ QED.

$\forall A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$ $\quad E(E_n X) = EX$

$\Omega \in \mathcal{F}_n$. Put $A = \Omega$. $\Rightarrow \displaystyle\sum_{\omega \in \Omega} E_n X(\omega) p(\omega) = \sum_{\omega \in \Omega} X(\omega) p(\omega)$

Question 5.41. *Conversely, if $\boldsymbol{E}M_n$ is constant, is M a martingale?* NO!! (IOU Eg there are many).

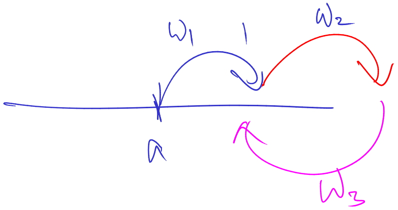
Eg 0: $M_n = 0$ $\forall n$ is a mg.

Example 5.42. Unbiased random walks are martingales.

Say $\omega_i \longrightarrow$ outcome of a fair coin $\begin{cases} \omega_i = 1 & \text{w/ prob } 1/2 \\ \omega_i = -1 & \text{"} \quad 1/2. \end{cases}$

$\omega = (\omega_1, \dots, \omega_N) \longleftarrow N$ iid fair coins.

$X_0 = a \in \mathbb{R}$. $\qquad X_{n+1}(\omega) = X_n(\omega) + \omega_{n+1} = a + \sum_{k=1}^{n+1} \omega_k$



Claim: X_n is a mg.

Pf: (0) X adapted (Yes)

(1) $E_n X_{n+1} = E_n(X_n + \omega_{n+1}) \overset{\text{lin.}}{=} \underbrace{E_n X_n}_{X_n} + E_n \omega_{n+1}$

$$= X_n + E\omega_{n+1} \qquad (\because \omega_{n+1} \text{ ind of } \mathcal{F}_n)$$

$$(\because X_n \text{ is } \mathcal{F}_n\text{- meas})$$

$$= X_n + 0$$

QED

Example 5.43. More generally, if $M_{n+1} - M_n$ is mean 0 and independent of $\mathcal{F}_n$, then M is a martingale. ← (indep increments)

Question 5.44. *If M is a martingale, must $M_{n+1} - M_n$ be independent of $\mathcal{F}_n$?* NO! M mg $\not\Rightarrow$ $M_{n+1} - M_n$ is indep of $\mathcal{F}_n$. HW3 Q2

Pf: ① Assume $M_{n+1} - M_n$ is ind of $\mathcal{F}_n$
② $E(M_{n+1} - M_n) = 0$
$\Big\} \Rightarrow M$ is a mg.

Pf: $E_n M_{n+1} = E_n(M_{n+1} - M_n + M_n) = \underbrace{E_n(M_{n+1} - M_n)}_{\substack{\| \\ E(M_{n+1}-M_n) \text{ (} \because \text{ ind)} \\ \| \\ 0}} + \underbrace{E_n M_n}_{\substack{\| \\ M_n \\ \text{(adapted)}}}$ QED

Markov: $E_n \cancel{f}(X_{n+1}) = \cancel{g}(X_n)$

Last time: Martingales $\longrightarrow$ M is a mg if $\underbrace{E_n M_{n+1}} = M_n$

(fair game)

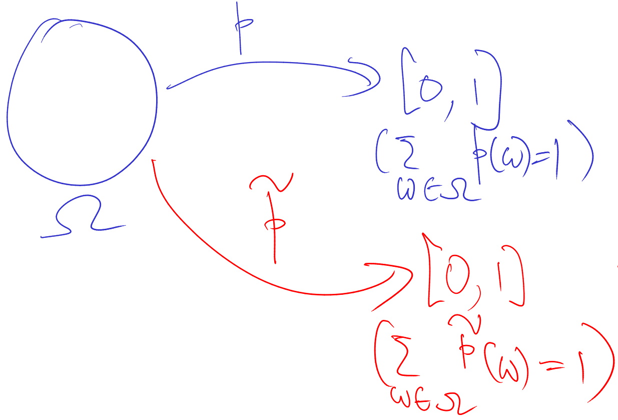
5.5. Change of measure.

- Let $p \colon \Omega \to [0,1]$ be a probability mass function on Ω, and $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$ be the probability measure.
- Let $\tilde{p} \colon \Omega \to [0,1]$ be another probability mass function, and define a second probability measure $\tilde{\boldsymbol{P}}$ by $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$.

Definition 5.47. We say $\boldsymbol{P}$ and $\tilde{P}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Remark 5.48. When Ω is finite, $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if and only if we have $p(\omega) = 0 \iff \tilde{p}(\omega) = 0$ for all $\omega \in \Omega$.

We let $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ denote the expectation and conditional expectations with respect to $\tilde{\boldsymbol{P}}$ respectively.



$A \subseteq \Omega \quad P(A) = \sum_{\omega \in A} p(\omega)$

$A \subseteq \Omega, \quad \tilde{P}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$

① X a RV. $\qquad EX = \sum_{\omega \in \Omega} X(\omega)\, p(\omega)$ (Expected value under P)

② Given the new PMF $\tilde{p}$, define $\tilde{E}X = \sum_{\omega \in \Omega} X(\omega)\, \tilde{p}(\omega)$ (Expected val under $\tilde{P}$)

③ $E_n X = E(X \mid \mathcal{F}_n)$ = cond exp of X given $\mathcal{F}_n$

ⓐ $E_n X$ is an $\underline{\mathcal{F}_n}$ meas RV.

ⓑ $\forall A \in \underline{\mathcal{F}_n}$ $\quad \sum_{\omega \in A} E_n X(\omega)\, p(\omega) = \sum_{\omega \in A} \underline{X(\omega)}\, p(\omega)$

④ $\tilde{E}_n X \longrightarrow \tilde{E}(X \mid \mathcal{F}_n) \longrightarrow$ cond exp of X given $\mathcal{F}_n$

ⓐ $\tilde{E}_n X$ is an $\mathcal{F}_n$-meas RV

ⓑ $\forall A \in \mathcal{F}_n$, $\quad \sum_{\omega \in A} \tilde{E}_n X(\omega)\, \tilde{p}(\omega) = \sum_{\omega \in A} X(\omega)\, \tilde{p}(\omega)$

Example 5.49. Let Ω be the sample space corresponding to N i.i.d. fair coins (heads is 1, tails is -1). Let $a \in \mathbb{R}$ and define $X_{n+1}(\omega) = X_n(\omega) + \omega_{n+1} + a$. For what a is there an equivalent measure $\tilde{P}$ such that X is a martingale?

$X_0 = 0$, $\quad X_1 = X_0 + \omega_1 + a$ $\qquad$ ($a \in \mathbb{R}$ same const).

$\qquad\qquad X_2 = X_1 + \omega_2 + a$

Q: Is X a mg? (Fair coin X is a mg $\iff a = 0$

$$E_n X_{n+1} = E_n(X_n + \omega_{n+1} + a) = X_n + \underbrace{E_n \omega_{n+1}}_{} + \overbrace{E_n a}^{a}$$

$$E_n X_{n+1} = X_n + 0 + \underline{\underline{a}}$$

$E\,\omega_{n+1} = 0$

(ω_{n+1} is indep of $\mathcal{F}_n$)

Goal: Find $\tilde{P}$ so that P & $\tilde{P}$ are equiv & X is a mg under $\tilde{P}$.

Let $\tilde{p}$ be a PMF (will find $\tilde{p}$ shortly)

$$\tilde{E}_n X_{n+1} = \tilde{E}_n\left(X_n + \omega_{n+1} + a\right) = X_n + \underbrace{\tilde{E}_n \omega_{n+1} + a}$$

Need to choose $\tilde{p}$ so that $\boxed{\tilde{E}_n \omega_{n+1} + a = 0}$

Say we choose $\tilde{p}$ so that the coins are iid &

$$\tilde{P}(\omega_n = 1) = \underline{\tilde{p}_1} \quad \& \quad \tilde{P}(\omega_n = -1) = \underline{\tilde{q}_1} = 1 - \tilde{p}_1.$$

Red Box $\to$ $\tilde{E}_n \omega_{n+1} + a = 0 \iff \tilde{E}\omega_{n+1} + a = \tilde{p}_1 1 + (1-\tilde{p}_1)(-1) + a$

$= 0$

$\iff 2\tilde{p}_1 - 1 + a = 0$

$$\boxed{\iff \tilde{p}_1 = \frac{1-a}{2}}$$

① need $\tilde{p}$ to be a PMF

② need $\tilde{P}$ equiv to P

Note $\tilde{p}_1 \in (0,1) \iff a \in (-1,1)$

Let $\tilde{P}$ be the Pr meas where each coin comes up heads with prob. $\tilde{p}_1 = \frac{1-a}{2} \in (0,1)$

Then X is a mg under $\tilde{P}$!

$(p(\omega) = 0 \iff \tilde{p}(\omega) = 0)$

$\tilde{p}(\omega) = \tilde{p}(\omega_1, \omega_2 \cdots \omega_N) = (\tilde{p}_1)^{\#\text{heads}} (1-\tilde{p}_1)^{\#\text{tails}}$

Note $p_1 \in (0,1) \iff \tilde{P}$ & P are eq

Example 5.50. Suppose now $\boldsymbol{P}(\omega_n = 1) = p$ and $\boldsymbol{P}(\omega_n = -1) = q = 1 - p$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $D_n = (1+r)^{-n}$ be the "discount factor". Find an equivalent measure under which $D_n S_n$ is a martingale.

Last time: Change of measure:

$\Omega \quad p$ (PMF) $\qquad P(A) = \sum_{\omega \in A} p(\omega)$

$\hookrightarrow$ (input) $\tilde{p}$ (new PMF) $\qquad \tilde{P}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$

① P & $\tilde{P}$ are equiv: if $P(A) = 0 \iff \tilde{P}(A) = 0$

(alternately $p(\omega) = 0 \iff \tilde{p}(\omega) = 0$).

$X_{n+1} = X_n + \underbrace{\omega_{n+1}}_{\pm 1} + a \qquad (a \in \mathbb{R}) \rightarrow a \neq 0 \Rightarrow X$ is not a mg under P

Find $\tilde{P}$ under which X is a mg!

Example 5.50. Suppose now $\boldsymbol{P}(\omega_n = 1) = p$ and $\boldsymbol{P}(\omega_n = -1) = q = 1 - p$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $D_n = (1+r)^{-n}$ be the "discount factor". Find an equivalent measure under which D_nS_n is a martingale.

$$(\text{Value of cash at time } n) \cdot \underbrace{D_n}_{(1+r)^{-n}} = \text{value of cash at time } 0$$

Q: Find $\tilde{P}$ s.t. $\tilde{P}$ is equiv to P & $\underline{D_n} S_n$ is a mg.

Say unk $\tilde{P}$ $\to$ $\tilde{P}(\omega_i = 1) = \tilde{p}_1$

& $\tilde{P}(\omega_i = -1) = 1 - \tilde{p}_1 = \tilde{q}_1$

Compute $\tilde{E}_n(D_{n+1} S_{n+1})$ $\quad$ (Want $= D_n S_n$)

$$X_n = X(\omega) = \begin{cases} u & \text{if } \omega_n = 1 \\ d & \text{if } \omega_n = -1 \end{cases} \qquad (\omega = (\omega_1, \omega_2 \dots \omega_N))$$

Then $S_{n+1} = X_{n+1} S_n$

$$\Rightarrow \tilde{E}_n (D_{n+1} S_{n+1}) = (1+r)^{-(n+1)} \tilde{E}_n (X_{n+1} S_n)$$

$$= D_{n+1} S_n \tilde{E}_n X_{n+1} \qquad (\because S_n \text{ is } \mathcal{F}_n \text{ meas})$$

$$= D_{n+1} S_n \tilde{E} X_{n+1} \qquad (\because X_{n+1} \text{ is ind of } \mathcal{F}_n \text{ under } \tilde{P})$$

$M_n = D_n S_n$

$$= D_{n+1} S_n \left(u \tilde{p}_1 + d(1-\tilde{p}_1) \right) \overset{\text{Want}}{=} D_n S_n$$

Choose $\tilde{p}_1$ so that $(1+r)^{-(n+1)} \cancel{S_n} \left(u \tilde{p}_1 + d \tilde{q}_1 \right) = (1+r)^{-n} \cancel{S_n}$

$$\boxed{\Rightarrow \quad u \tilde{p}_1 + d \tilde{q}_1 = 1+r}$$

Solve: $\tilde{p}_1 (u-d) + d = 1+r \iff \boxed{\tilde{p}_1 = \dfrac{(1+r) - d}{u-d}}$

Will only give an equiv meas $\iff d < 1+r < u$

6. The multi-period binomial model

Example 6.1 (Binomial model revisited). Assume $\Omega = \{\pm 1\}^N$. Let $u, d > 0$, $S_0 > 0$. Define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1, \\ dS_n & \omega_{n+1} = -1. \end{cases}$

- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p\boldsymbol{P}(\omega_n = 1) > 0$ and $q = 1 - p = \boldsymbol{P}(\omega_n = -1) > 0$.
- Let $r > -1$ be the interest rate, and $D_n = (1+r)^{-n}$ be the discount factor.

Theorem 6.2. *There exists a (unique) equivalent measure* $\tilde{\boldsymbol{P}}$ *under which process* $\boxed{D_n S_n}$ *is a martingale* if and only if $d < 1 + r < u$. *In this case* $\tilde{\boldsymbol{P}}$ *is given by:*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p} = \frac{1 + r - d}{u - d}, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q} = \frac{u - (1+r)}{u - d}.$$

Definition 6.3. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.4. If there are more than one risky assets, $S^1, \ldots, S^k$, then we require $D_n S_n^1, \ldots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{P}$.

Find $\tilde{P}$ done.

$D_n S$

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.*
- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

Theorem 6.5. *The discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.6. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

6. The multi-period binomial model

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $u, d > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1, \\ dS_n & \omega_{n+1} = -1. \end{cases}$
- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p_1 = \boldsymbol{P}(\omega_n = 1) > 0$ and $q_1 = 1 - p_1 = \boldsymbol{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $r > -1$.
- $D_n = (1 + r)^{-n}$ be the discount factor (\$1 at time n is worth $\$D_n$ at time 0.)

Theorem 6.1. *There exists a (unique) equivalent measure* $\tilde{\boldsymbol{P}}$ *under which process* $\boxed{D_n S_n}$ *is a martingale* if and only if $\boxed{d < 1 + r < u.}$ *In this case* $\tilde{\boldsymbol{P}}$ *is the probability measure obtained by tossing* N *i.i.d. coins with*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p}_1 = \frac{1 + r - d}{u - d}, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q}_1 = \frac{u - (1 + r)}{u - d}.$$

Definition 6.2. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.3. If there are more than one risky assets, $S^1, \ldots, S^k$, then we require $D_n S_n^1, \ldots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.* (Δ_n must be $\mathcal{F}_n$ meas)
- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

Definition 6.4. A *self-financing portfolio* is a portfolio whose wealth evolves according to (X_n is adapted)

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,$$

for some adapted process Δ_n.

Theorem 6.5. *Let $d < 1 + r < u$, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.6. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

Before proving Theorem 6.5, we consider a few consequences:

Theorem 6.7. *The multi-period binomial model is arbitrage free if and only if* $\boxed{d < 1 + r < u}$.

Remark 6.8. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

$\Rightarrow$ Pf: 1 Market has no arbitrage if $X_0 = 0$

$X_n \to$ wealth at time n of a self-fincng portfolio.

$X_0 = 0$, $\underbrace{X_n \geq 0}_{\text{no risk}}$ (almost surely) $\Rightarrow \underline{X_N = 0}$ (almost surely)

2 Pf: If $d \geq 1+r$ or $u \leq 1+r \Rightarrow$ markt has arb. (hw 2)

3 Say X_n is a the wealth of a self financing portfolio.

Suppose $X_N \geq 0$ almost surely & $X_0 = 0$

NTS $X_N = 0$.

Know $D_n X_n$ is a mg under $\tilde{P}$.

(∵ mg's have const exp) $\Rightarrow$ $\tilde{E}\, D_N X_N = \tilde{E}\, D_0 X_0 = 0$

$\Rightarrow$ $\tilde{E}(1+r)^{-N} X_N = 0$

Note $(1+r)^{-N} X_N \geq 0$

$\Rightarrow (1+r)^{-N} X_N = 0$ almost surely

$\Rightarrow X_N = 0$ almost surely QED.

Only used: If X_n is self-fin then $D_n X_n$ is a mg under $\tilde{P}$

(Risk Neutral pricing formula)

Theorem 6.9. *Let $d < 1 + r < u$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays V_N at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_N V_N)\,. = \tilde{E}_n\left(\frac{D_N V_N}{D_n}\right) \quad (\because D_n \text{ is } \mathcal{F}_n\text{-meas})$$

Pf: $D_n V_n = \boxed{\tilde{E}_n(D_N V_N)}$

Claim: Let $M_n = \tilde{E}_n(D_N V_N)$. Q: Is M a mg under $\tilde{P}$?

Claim: M is a mg under $\tilde{P}$. (Pf: $\tilde{E}_n M_{n+1} = \tilde{E}_n(\tilde{E}_{n+1} D_N V_N)$

$\overset{\text{tower}}{=} \tilde{E}_n(D_N V_N) = M_n$. QED)

Let $V_n = \frac{1}{D_n}\tilde{E}_n(D_N V_N) \Rightarrow \underbrace{D_n V_n}_{} = \underbrace{\tilde{E}_n(D_N V_N)}_{} = M_n$ (mg under $\tilde{P}$)

By Thm 6.5 : $\underline{\underline{V_n}}$ is the wealth of a self financing portfolio.

$V_N \longrightarrow$ Payoff of my security $\Rightarrow$ V_n = wealth at time n of a R. Portfolio

$\Rightarrow$ (Wealth of R. Portfolio) V_n = AFP of the security.

QED.

(Note : in this thm we used if $D_n X_n$ is a mg under $\tilde{P}$, then we must have X_n = wealth of a self financing portfolio).

IOU : Pf of Thm 6.5.

Remark 6.10. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{u - d} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{u - d}$$

Prop: If M is a mg then $EM_n = EM_0 \; \forall n$.

$\hookrightarrow$ Pf: Claim $EM_n = EM_{n+1}$ (& use induction)

Know $M_n = \underline{E_n M_{n+1}} \Rightarrow EM_n = \underbrace{E E_n}_{} M_{n+1}$

$E E_n X = EX$

EM_{n+1} QED.

Recall: Binom model

$$S_{n+1} \begin{cases} u\, S_n & \omega_{n+1} = 1 \\ d\, S_n & \omega_{n+1} = -1 \end{cases}$$

& $d < 1+r < u$ ($r \to$ int rate)

$\tilde{P} \to$ RNM. $\tilde{P}(\omega_i = 1) = \dfrac{1+r-d}{u-d} \in (0,1)$

$\tilde{P}(\omega_i = -1) = \dfrac{u-(1+r)}{u-d} \in (0,1)$.

$\to \tilde{E}_n(D_{n+1} S_{n+1}) = D_n S_n$

Thm (IOU): X_n is the wealth of a self fin Portfolio $\iff D_n X_n$ is a mg under $\tilde{P}$.

Theorem 6.9. *Let $d < 1 + r < u$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays* $\boxed{V_N}$ *at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$\boxed{V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_N V_N).}$$

$\left(n=0: \quad V_0 = \tilde{E}(\cancel{D_0} D_N V_N) \right)$

Pf: Let $X_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N)$.

① $X_N = \frac{1}{D_N} \tilde{E}_N (D_N V_N) = V_N$.

② Note $D_n X_n = \tilde{E}_n (D_N V_N)$. Tower prop $\Rightarrow$ $D_n X_n$ is a mg under $\tilde{P}$

$\Rightarrow$ (By thm) X_n = wealth of a self fin Portfolio at time n.

③ X_n = wealth of a Replicating Portfolio at time n.

$\Rightarrow X_n = \underbrace{AFP}_{V_n.}$ $\Rightarrow$ QED.

Only used $D_n X_n$ a $\tilde{P}$ mg $\Rightarrow X$ self fin.

Remark 6.10. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$\to \omega = (\omega_1, \omega_2, \cdots \omega_N)$

$$\underbrace{\Delta_n(\omega)}_{\text{\# shares of stock at time } n} = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u-d)(S_n(\omega))} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u-d)(S_n(\omega))}$$

$\omega' = (\omega_1 \cdots \omega_n)$

$\omega'' = (\omega_{n+2}, \cdots \omega_N)$

Let $X_n = V_n$ —

Holdings at time n $\begin{cases} \to \Delta_n \text{ shares of stock} \\ \to (X_n - \Delta_n S_n) \text{ cash.} \end{cases}$

Self fin $\Rightarrow$ $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

$X_{n+1}(\omega) \longrightarrow$ only dep on 1^{st} $n+1$ coin tosses.

Write $X_{n+1}(\omega) = X_{n+1}(\omega', \omega_{n+1}, \cancel{\omega''}) = X_{n+1}(\omega', \omega_{n+1})$.

$$X_{n+1}(\omega', \omega_{n+1}) = \Delta_n(\omega') S_{n+1}(\omega', \omega_{n+1}) + (1+r)\left(X_n(\omega') - \Delta_n(\omega') S_n(\omega')\right)$$

2 Eq: $\omega_{n+1} = 1$ & $\omega_{n+1} = -1$

$$\begin{cases} X_{n+1}(\omega', 1) = \Delta_n(\omega') S_n(\omega')\, u + (1+r)\left(X_n(\omega') - \Delta_n(\omega') S_n(\omega')\right) \\ X_{n+1}(\omega', -1) = \;'' \quad '' \quad d + \;'' \quad '' \quad '' \quad '' \end{cases}$$

subtract: $X_{n+1}(\omega', 1) - X_{n+1}(\omega', -1) = \Delta_n(\omega') S_n(\omega')(u-d)$

$(X_n = V_n)$

Proof of Theorem 6.5 part 1. Suppose X_n is the wealth of a self-financing portfolio. Need to show $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.

Assume $d < 1+r < u$. $\tilde{P} \to$ RNM. $\left(\tilde{E}_n (S_{n+1} D_{n+1}) = D_n S_n \right)$

$D_n = (1+r)^{-n}$

$X \to$ self fin.

$$X_{n+1} = \Delta_n S_{n+1} + (X_n - \Delta_n S_n)(1+r)$$

$\boxed{\text{NTS } D_n X_n \text{ is a } \tilde{P} \text{ mg}}$

$$\Rightarrow D_{n+1} X_{n+1} = \Delta_n D_{n+1} S_{n+1} + (1+r) D_{n+1} (X_n - \Delta_n S_n)$$

$$\Rightarrow \tilde{E}_n (D_{n+1} X_{n+1}) = \Delta_n \tilde{E}_n (D_{n+1} S_{n+1}) + D_n (X_n - \Delta_n S_n)$$

$$= \Delta_n \cancel{D_n} S_n + D_n (X_n - \cancel{\Delta_n S_n}) = D_n X_n \quad \text{QED.}$$

$$= p_n X_n$$

Proof of Theorem 6.5 part 2. $\boxed{\text{Suppose } D_n X_n \text{ is a martingale under } \tilde{\boldsymbol{P}}.}$ Need to show $\boxed{X_n \text{ is the wealth of a self-financing portfolio.}}$

$$X_{n+1}(\omega) = X_{n+1}(\omega', \omega_{n+1}, \omega'') \qquad \left(\omega' = (\omega_1 \dots \omega_n) \quad \omega'' = (\omega_{n+2}, \dots \omega_N)\right)$$

$$= X_{n+1}(\omega', \omega_{n+1})$$

$$\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} \in \mathbb{R}^2 \quad (\text{same vector}).$$

Write as a L.C. of $\begin{pmatrix} u \\ d \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Can always write $\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} = \underbrace{\Delta_n(\omega')\, S_n(\omega') \begin{pmatrix} u \\ d \end{pmatrix}} + \Gamma_n(\omega') \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow X_{n+1}(\omega', \omega_{n+1}) = \Delta_n(\omega')\, S_{n+1}(\omega', \omega_{n+1}) + \Gamma_n(\omega').$$

$\Rightarrow$ Can always write $X_{n+1} = \Delta_n S_{n+1} + \underline{\Gamma_n}$ for some $\underline{\text{adapted}}$ processes Δ & Γ.

$$\boxed{\text{NTS}: \Gamma_n = (X_n - \Delta_n S_n)(1+r)}$$

Pf: Note $D_{n+1} X_{n+1} = \Delta_n D_{n+1} S_{n+1} + D_{n+1} \Gamma_n$

Know $D_n X_n$ is a $\tilde{P}$ mg.

$$\Rightarrow D_n X_n = \tilde{E}_n (D_{n+1} X_{n+1}) = \tilde{E}_n (RHS)$$

$$= \Delta_n (D_n S_n) + D_{n+1} \Gamma_n .$$

$$\Rightarrow \Gamma_n = \frac{1}{D_{n+1}} \left(D_n X_n - \Delta_n D_n S_n \right) = (1+r) X_n - \Delta_n S_n (1+r)$$

$$= (1+r)(X_n - \Delta_n S_n) \quad \text{QED.}$$

$$\begin{tikzcd}[row sep=small] & {} \\ S^1 \arrow[ur, "u=2"] \arrow[dr, "d=\frac{1}{2}"'] & \\ & {} \end{tikzcd}$$

(Prob Heads = 90%)

(Prob Tails = 10%)

$C^1 \to$ call option on S^1

$$\begin{tikzcd}[row sep=small] & {} \\ S^2 \arrow[ur, "u=2"] \arrow[dr, "d=\frac{1}{2}"'] & \\ & {} \end{tikzcd}$$

$P(\text{Heads}) = 99\%$

$P(\text{Tails}) = 1\%$

$C^2 \to$ call option on S^2

$S_0^1 = S_0^2$ | Q: $C_0^1 = C_0^2$

7. State processes.

$\to u, d, r \quad 0 < d < 1+r < u.$

Question 7.1. *Consider the N-period binomial model, and a security with payoff $\boxed{V_N}$. Let X_n be the arbitrage free price at time $n \leqslant N$, and Δ_n be the number of shares in the replicating portfolio. What is an algorithm to find X_n, Δ_n for all $n \leqslant N$? How much is the computational time?*

(1) Know $\tilde{P}$: $\quad \tilde{p} = \dfrac{1+r-d}{u-d} \qquad \tilde{q} = \dfrac{u-(1+r)}{u-d}.$

$\hookrightarrow \tilde{P}(\omega_i = +1) = \tilde{p}$ & $\tilde{P}(\omega_i = -1) = \tilde{q}$ (RNM).

$X_N = V_N \qquad (X_n = \text{AFP} = \text{wealth of Rep Portfolio}).$

$D_n X_n$ is a $\tilde{P}$ mg. $\qquad (D_n = (1+r)^{-n})$

$\Rightarrow \tilde{E}_n (D_{n+1} X_{n+1}) = D_n X_n$

$$\Rightarrow \quad X_n = \frac{\tilde{E}_{n} X_{n+1}}{1+r}$$

$$X_n = X_n(\omega) \qquad \text{(Remark } X_n \text{ only dep on } \omega_1, \cdots \omega_n$$

$$\& \ \omega_{n+1} \text{ is indep of } \omega_1, \cdots \omega_n)$$

$$X_n(\omega) = X_n(\omega_1, \omega_2, \cdots \omega_n) = \frac{\tilde{E}_n\left(X_{n+1}(\omega_1, \omega_2, \cdots, \omega_n, \omega_{n+1})\right)}{1+r}$$

$$\stackrel{\text{(Ind lemma)}}{=} \frac{1}{1+r}\left[\tilde{p}\, X_{n+1}(\omega_1, \cdots \omega_n, 1) + \tilde{q}\, X_{n+1}(\omega_1, \omega_2 \cdots \omega_n, -1)\right]$$

$\Rightarrow$ To find X_N : Given $X_N = V_N$ (Given)

$$X_n(\omega_1, \omega_2, \cdots \omega_n) = \frac{1}{1+r}\left(\tilde{p}\, X_{n+1}(\omega_1, \cdots \omega_n, 1) + \tilde{q}\, X_{n+1}(\omega_1, \cdots \omega_n, -1)\right)$$

Backward Ind

For Δ_n : Last time

$$\Delta_n(\omega) = \frac{X_{n+1}(\omega_1, \cdots \omega_n, 1) - X_{n+1}(\omega_1, \cdots \omega_n, -1)}{(u-d)\, S_n(\omega)}$$

Computational Cost : $\approx O(2^N) \leftarrow$ Practical ?

(time)

Eg: $N = 100$. Cost $\approx 2^{100}$ operations

$$2^{10} \approx 10^3 \Rightarrow 2^{100} \approx \underset{\sim}{10^{30}}$$

Guess: 10 bill op per sec. $\approx 10^{10}$ op per sec.

total time $\approx 10^{20}$ <u>seconds</u> (life of universe $\approx 10^{19}$ sec)

Goal $\longrightarrow$ Improve this. (Can do it in $\boxed{\cancel{O(N)}}$ or $O(N^2)$ time)

(Binom model)

Theorem 7.2. *Suppose a security pays $V_N = g(S_N)$ at maturity N for some (non-random) function g. Then the arbitrage free price at time $n \leqslant N$ is given by $V_n = f_n(S_n)$, where:*

(1) $f_N(x) = V_N(x)$ *for* $x \in \text{Range}(S_N)$. $g(x)$

(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \text{Range}(S_n)$.

Remark 7.3. Reduces the computational time from $\boxed{O(2^N)}$ to $O(\sum_0^N |\boxed{\text{Range}(S_n)}|) = \boxed{O(N^2)}$ for the Binomial model.

Remark 7.4. Can solve this to get $f_n(x) = \displaystyle\sum_{k=0}^{N-n} \binom{N-n}{k} f_N(xu^k d^{N-n-k})$

Say $X_{n+1} = f_{n+1}(S_{n+1})$. (True for $n+1 = N$)

Know $X_n = X_n(\omega_1, \cdots \omega_n) = \frac{1}{1+r}\left(\tilde{p}\, X_{n+1}(\omega_1, \cdots \omega_n, +1) + \tilde{q}\, X_{n+1}(\omega_1, \cdots \omega_n, -1)\right)$

$= \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(S_{n+1}(\omega_1 \cdots \omega_n, +1)) + \tilde{q}\, f_{n+1}(S_{n+1}(\omega_1, \cdots \omega_n, -1))\right)$

$$\Rightarrow X_n(\omega) = \underbrace{\frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(S_n(\omega)\, u) + \tilde{q}\, f_{n+1}(S_n(\omega)\cdot d)\right)}_{\text{some fn of } S_n\,!}$$

Let $x = S_n(\omega)$.

$$\boxed{\text{Let } f_n(x) = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(ux) + \tilde{q}\, f_{n+1}(dx)\right)}$$

done!

Indep lemma: $X \to \mathcal{F}_n$ meas

$Y \to$ ind of $\mathcal{F}_n$.

$$E_n f(X, \underline{Y}) = \sum_{1}^{k} f(X, \underline{y_i}) \cdot \underbrace{P(Y = y_i)}$$

Last time:

RNP formula: $V_n = \frac{1}{P_n} \tilde{E}_n(D_N V_N)$ ← not practical to compute with

(Computational time $\to 2^N$)

Theorem 7.2. *Suppose a security pays $V_N = g(S_N)$ at maturity N for some (non-random) function g. Then the arbitrage free price at time $n \leqslant N$ is given by $V_n = f_n(S_n)$, where:*

(1) $f_N(x) = g(x)$ *for* $x \in \text{Range}(S_N)$.

(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \text{Range}(S_n)$.

Remark 7.3. Reduces the computational time from $O(2^N)$ to $O(\sum_0^N |\text{Range}(S_n)|) = O(N^2)$ for the Binomial model.

Remark 7.4. Can solve this to get $f_n(x) = \displaystyle\sum_{k=0}^{N-n} \binom{N-n}{k} f_N(xu^k d^{N-n-k}) \cdot \frac{1}{(1+r)^{N-n}} \tilde{p}^k \tilde{q}^{N-n-k}$

① Compute $f_{N-1}(x) = \dfrac{1}{1+r}\left(\tilde{p} f_N(ux) + \tilde{q} f_N(dx)\right)$

② Compute $f_{N-2}(x) = \dfrac{1}{1+r}\left(\tilde{p} f_{N-1}(ux) + \tilde{q} f_{N-1}(dx)\right)$

$$= \frac{1}{1+r}\left(\tilde{p}\left(\frac{1}{1+r}\left(\tilde{p} f_N(u^2x) + \tilde{q} f_N(udx)\right)\right) + \tilde{q}\left(\frac{1}{1+r}\left(\tilde{p} f_N(udx) + \tilde{q} f_N(d^2x)\right)\right)\right)$$

$$= \frac{1}{(1+r)^2}\left[\tilde{p}^2\, f_N(u^2 x) + 2\tilde{p}\tilde{q}\, f_N(u d x) + \tilde{q}^2 f_N(d^2 x)\right]$$

③ Iterate this & get Rule 7.4.

Question 7.5. *How do we handle other securities? E.g. Asian options (of the form $g(\sum_0^N S_k)$)?*

European call options → payoff $(S_N - K)^+$

Asian call option → payoff $\left(\frac{1}{N+1}\sum_0^N S_k - K\right)^+$ ← Now

(American call → can exercise at any time $\leq N$) ← IOU

Definition 7.6. We say a process X is a *Markov process* if $\boldsymbol{P}(X_{n+1} = x_{n+1} \mid X_1 = x_1, \ldots, X_n = x_n) = \boldsymbol{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$.

Theorem 7.7. *A process X is Markov if and only if for every (bounded, continuous) function f, there exists a function g such that $\boldsymbol{E}_n f(X_{n+1}) = g(X_n)$.*

Question 7.8. *If X_n represents i.i.d. coin tosses, is X_n Markov? Is $Y_n = \sum_0^n X_k$ Markov?*

$\forall f$

$$E_n f(X_{n+1}) = E\left(f(X_{n+1}) \mid \mathcal{F}_n \right) \overset{\text{Markov}}{=} g(X_n)$$

$$E_n f(X_{n+1}) \overset{\text{indep}}{=} E f(X_{n+1}) \overset{\text{iid}}{=} E f(X_1) \quad (\text{some } \#)$$

$$(\text{Chose } g(x) = E f(X_1) \ \forall x)$$

$X_n \to$ iid. $\quad Y_n = \sum_0^n X_k \quad (Y_0 = 0)$

Q: Is Y markov $\begin{cases} \text{Guess} \to \text{No} \\ \text{Guess} \to \text{Yes!} \end{cases}$

Note: $Y_{n+1} = Y_n + X_{n+1}$

Compute $E_n f(Y_{n+1}) = E_n f(\underline{Y_n} + X_{n+1})$

$$\underset{\text{indep lem}}{=} \sum f(\underbrace{Y_n + x_i}) \underbrace{P(X_1 = x_i)}_{\text{fix \#'s.}}$$

$= \text{some fn of } Y_n. \Rightarrow \boxed{Y \text{ is Markov}}$

Question 7.9. *Is S_n (stock in the Binomial model) Markov under $\tilde{\boldsymbol{P}}$? Is $A_n = \frac{1}{n}\sum_0^n S_k$ Markov under $\tilde{\boldsymbol{P}}$?*

Guess S_n is markov.

Let $X_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$

$$\tilde{E}_n f(S_{n+1}) = \tilde{E}_n f(X_{n+1} S_n)$$

$$\underset{\text{indep lemma}}{=} \tilde{p} f(u S_n) + \tilde{q} f(d S_n)$$

$$= \text{Same fn of } S_n. \quad \Rightarrow S_n \text{ is Markov.}$$

Claim: A_n is NOT Markov

(Intuition $\rightarrow$ To find A_{n+1} need to know S_{n+1}

depends on S_n

Can't determine S_n in terms of A_n.)

Question 7.10. *Is* (S_n, A_n) *Markov?* (True $\rightarrow$ IOU)

Definition 7.11. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n. $(n \leq m)$

Remark 7.12. For state processes given f_N, we find f_n by backward induction. The number of computations at time n is of order $\text{Range}(Y_n)$.

Remark 7.13. The fact that S_n is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process. (Binomial model $0<d<1+r<u$)

Pf of rmk 7.13: Consider a sec with payoff $f_N(S_N)$

Let $n \leq N$. AFP at $n = V_n = \frac{1}{D_n} \tilde{E}_n D_N f_N(S_N)$

$n = N-1$: AFP $= V_{N-1} = \frac{1}{(1+r)} \tilde{E}_{N-1} f_N(S_N)$

$\overset{\text{Markov}}{=} \frac{1}{1+r}$ some fn (S_{N-1}) state at time $N-1$ (not history)

Last time: Markov Process: "Memoryless"

$$E_n\, g(X_{n+1}) = g(X_n)$$

$$\Leftrightarrow P\left(X_{n+1} = x_{n+1} \,\middle|\, \boxed{X_1 = x_1,\ X_2 = x_2,\ \ldots}\ X_n = x_n\right)$$

$$= P\left(X_{n+1} = x_{n+1} \,\middle|\, X_n = x_n\right)$$

Definition 7.11. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n.

Remark 7.12. For state processes given f_N, we find f_n by backward induction. The number of computations at time n is of order $\text{Range}(Y_n)$.

Remark 7.13. The fact that S_n is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process. (last time).

Y^i Super script $\longrightarrow$ coordinates of the process

Y^i_n Sub script $\longrightarrow$ time

$\|$ ith coordinate of Y at time n.

Notation for vector valued processes:

$$Y_n = (Y_n^1, Y_n^2, \ldots Y_n^d) = \begin{pmatrix} Y_n^1 \\ Y_n^2 \\ \vdots \\ Y_n^d \end{pmatrix}$$

Q: $$E Y_n \stackrel{\text{def}}{=} (E Y_n^1, E Y_n^2, \ldots E Y_n^d) = \begin{pmatrix} E Y_n^1 \\ E Y_n^2 \\ \vdots \\ E Y_n^d \end{pmatrix}$$

($Y_n^2 = 2^{\text{nd}}$ coordinate)

$(Y_n^2)^2 =$ square.

Similarly $E_m Y_n =$ cond exp

$$= \begin{pmatrix} E_m Y_n^1 \\ \vdots \\ E_m Y_n^d \end{pmatrix}$$

(N period Binomial, $0 < d < 1+r < u$)

Theorem 7.14. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process. Suppose we can find functions $g_1, \dots, g_N$ such that $Y_{n+1}(\omega) = g_{n+1}(Y_n(\omega), \omega_{n+1})$. Then Y is a state process.*

If $Y_{n+1}(\omega) = g_{n+1}(Y_n(\omega), \omega_{n+1})$ $\quad g_{n+1} \to$ non random fn.

Pf: Let $m \leq N$. Consider a sec that pays $f_m(Y_m)$ at maturity m.

Let $n \leq m$. NTS: AFP at time $n = f_n(Y_n)$ for some fn f.

Pf: Backward induction. (1) True for $n = m$.

(2) Assume True for $n+1$ (i.e. assume AFP at time $n+1 = f_{n+1}(Y_{n+1})$)

Prove for n. (i.e. NTS AFP at time $n = f_n(Y_n)$ for some fn n.

Pf of ②: (RNP: Sec pays V_N at mat N

then AFP at time $n = \frac{1}{D_n} \tilde{E}_n D_N V_N$)

Consider a security that pays $f_m(Y_m)$ at time m.

$\Rightarrow$ AFP at time $n = \frac{1}{D_n} \tilde{E}_n \left(D_m f_m(Y_m) \right)$

$$\Rightarrow \text{AFP at time } n = \frac{1}{D_n} \Big(\tilde{E}_n D_{n+1} \underbrace{(\text{AFP at time } n+1)}_{= f_{n+1}(Y_{n+1}) \text{ (ind Hyp)}} \Big)$$

$\Rightarrow$ AFP at time n = $\dfrac{D_{n+1}}{D_n} \tilde{E}_n f_{n+1}(Y_{n+1}) = \dfrac{1}{1+r} \tilde{E}_n f_{n+1}(Y_{n+1})$

$\frac{D_{n+1}}{D_n} \downarrow \frac{1}{1+r}$

$$= \frac{1}{1+r} \tilde{E}_n f_{n+1}\big(g_{n+1}(\underset{\mathcal{F}_n\text{-meas}}{Y_n}, \underset{\text{indep of } \mathcal{F}_n \text{ under } \tilde{P}}{\omega_{n+1}})\big)$$

indep lma

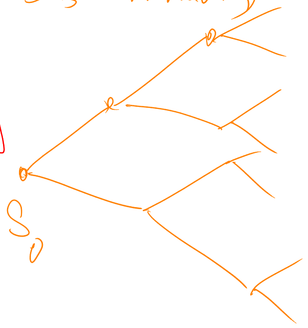
$$= \underbrace{\frac{1}{1+r}\Big(f_{n+1} \circ g_{n+1}(Y_n, 1)\, \tilde{p} + f_{n+1} \circ g_{n+1}(Y_n, -1)\, \tilde{q} \Big)}_{\text{some non-random fn of } Y_n.}$$

QED !!

Question 7.15. *Is $Y_n = S_n$ a state process?* ← Yes (because S is markov)

Question 7.16. *Is $Y_n = \max_{k\leqslant n} S_n$ a state process?* ← NO

Question 7.17. *Is $Y_n = (S_n, \max_{k\leqslant n} S_n)$ a state process?* ← Yes!!



Y_n adapted? → Yes!

Let $M_n = \max_{0 \le k \le n} S_k$.

$$Y_n = (S_n, M_n)$$

can write in terms of S_n & ω_{n+1}

$$M_{n+1} = \begin{cases} S_{n+1} & \text{if } S_{n+1} > M_n \\ M_n & S_{n+1} \le M_n \end{cases} =$$

$\Rightarrow$ Can express Y_{n+1} ~~as~~ as a fn of Y_n & ω_{n+1}

$\Rightarrow$ Y is a state process.

Question 7.18. *Let* $A_n = \sum_0^n S_k$. *Is* A_n *a state process?* NOT

Question 7.19. *Is* $Y_n = (S_n, A_n)$ *a state process?* ? ← Is

① Mg: $M_n = E_n M_{n+1} \iff \forall m \geq n, \; M_n = E_n M_m$

$$M_n = E_n M_{n+1} = E_n E_{n+1} M_{n+2} = E_n M_{n+2}$$

6.3. Options with random maturity. Consider the N period binomial model with $0 < d < 1 + r < u$.

Example 6.30 (Up-and-rebate option). Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Explicitly, let $\tau = \min\{n \leqslant N \mid S_n \geqslant U\}$, and let $\sigma = \tau \wedge N$. The up-and-rebate options pays $A\mathbf{1}_{\tau \leqslant N}$ at the random time σ.

inf

Remark 6.31. By convention $\min \emptyset = \infty$.

Up & rebate opt pays $A \mathbf{1}_{\{\tau \leq N\}}$ at time $\cancel{\tau}$ $\underbrace{\tau \wedge N}_{\sigma}$

$a \wedge b = \min\{a, b\}$

$a \vee b = \max\{a, b\}$

Definition 6.32. We say a random variable τ is a *stopping time* if:

(1) $\tau : \Omega \to \{0, \ldots, N\} \cup \infty$

(2) For all $n \leqslant N$, the event $\boxed{\{\tau \leqslant n\} \in \mathcal{F}_n}$.

Remark 6.33. We say τ is a finite stopping time if $\tau < \infty$ almost surely.

Remark 6.34. The second condition above is equivalent to requiring $\{\tau = n\} \in \mathcal{F}_n$ for all n.

$$\boxed{\forall n \quad \{\tau \leq n\} \in \mathcal{F}_n} \iff \forall n \quad \{\tau = n\} \in \mathcal{F}_n$$

Question 6.35. *Is $\tau = 5$ a stopping time?*

Question 6.36. *Is the first time the stock price hits U a stopping time?*

Question 6.37. *Is the last time the stock price hits U a stopping time?*

$\boxed{N = 370}$

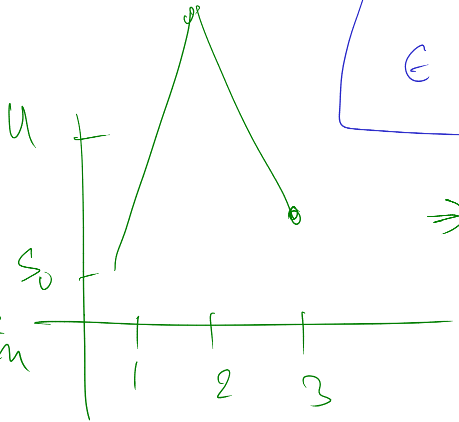
Yes: $\{\tau = n\} = \begin{cases} \Omega & n = 5 \\ \emptyset & n \neq 5 \end{cases} \in \mathcal{F}_0 \subseteq \mathcal{F}_n \ \forall n.$

$\tau = \min\{n \mid S_n \geq U\}$

$\{\tau \leq n\} = \cancel{\{S_n \geq U\}}$

$= \{S_0 \geq U\} \cup \{S_1 \geq U\} \cup \dots \cup \{S_n \geq U\} \in \mathcal{F}_n$

$= \{\max\{S_0, S_1 \dots S_n\} \geq U\} \in \mathcal{F}_n$



$\Rightarrow \{\tau \leq n\} \in \mathcal{F}_n \ \forall n$

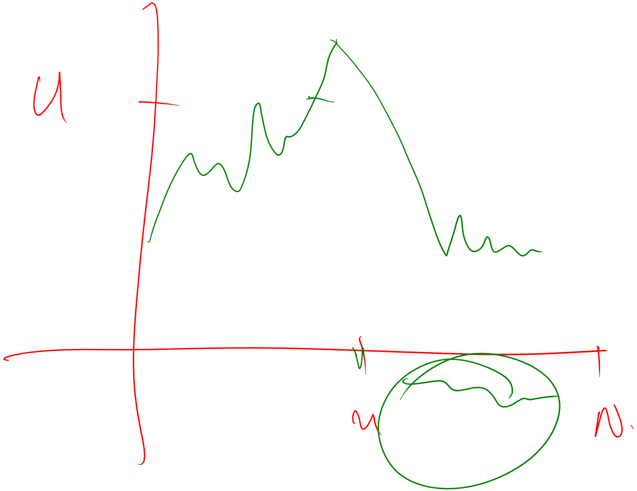
$\Rightarrow \tau$ is a stopping time.

Q: τ = last time S_n crosses U

NO! $\{\tau \le n\}$ involves knowing

$S_{n+1} \le U$, $S_{n+2} \le U$ ... etc.

$\notin \mathcal{F}_n$.



Question 6.38. *If σ and τ are stopping times, is $\sigma \wedge \tau$ a stopping time? How about $\sigma \vee \tau$?* ← Yes (use ~~union~~ intersection)

① $\sigma \wedge \tau : \Omega \longrightarrow \{0, \dots, n\} \cup \{\infty\}$ (Yes)

② NTS $\{\sigma \wedge \tau \leq n\} \in \mathcal{F}_n \quad \forall n.$

Pf: $\{\sigma \wedge \tau \leq n\} = \{\sigma \leq n\} \cup \{\tau \leq n\}$

$\{\omega \in \Omega \mid \sigma(\omega) \wedge \tau(\omega) \leq n\}$

$\{\sigma \leq n\} \in \mathcal{F}_n$ ($\because \sigma$ is a stopping time)

$\{\tau \leq n\} \in \mathcal{F}_n$ ($\because \tau$ is a stopping time).

σ, τ is a stopping time

Is $\sigma/2$ a stopping time

$\{\sigma/2 \leq n\} = \{\sigma \leq 2n\} \in \mathcal{F}_{2n}$ need not be in $\mathcal{F}_n$.

- Let G be an adapted process, and σ be a *finite* stopping time.
- Consider a derivative security that pays G_σ at the random time σ.
- ~~Note $G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\sigma=n}$.~~
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

$$G_\sigma = \sum_{n=0}^{N} \mathbb{1}_{\{\sigma = n\}} G_n$$

Definition 6.39. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$\boxed{X_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})}$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

$$G_\sigma(\omega) = G_{\sigma(\omega)}(\omega)$$

X_0 = initial wealth

Δ_n = # shares of stock at time n held in the portfolio.

$\underbrace{(X_0, (\Delta_n))}_{\text{trad strat.}}$

X_n = wealth at time n.

(RNP formula:
Payoff V_N at time N
the AFP at time n

$$= \frac{1}{D_n} \tilde{E}_n (D_N V_N)$$)

Last time: Securities with a random maturity time.

Payment time has to be a stopping time.

Recall: τ is a stopping time if

① $\tau : \Omega \longrightarrow \{0, \dots N\} \cup \{\infty\}$

& ② $\forall n \leq N$, $\{\tau \leq n\} \in \mathcal{F}_n$.

- Let G be an adapted process, and σ be a *finite* stopping time.
- Consider a derivative security that pays G_σ at the random time σ.
- Note $G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\sigma=n}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

$G_\sigma(\omega) = G_{\sigma(\omega)}(\omega)$

Definition 6.39. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$X_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Δ_n = # shares of stock held in the portfolio at time n.

$$X_n = \frac{1}{D_n} \tilde{E}(D_\sigma G_\sigma \mathbb{1}_{\sigma \geq n})$$

$\boxed{\text{when } n \leq \sigma}$

n

Pf: Let $X_n = \frac{1}{D_n} \tilde{E}_n(D_\tau G_\tau)$. Know X_n is $\mathcal{F}_n$-meas.

Claim: ① $D_n X_n$ is a mg under $\tilde{P}$

Pf: $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n\left(\tilde{E}_{n+1}(D_\tau G_\tau)\right) \overset{\text{tower}}{=} \tilde{E}_n(D_\tau G_\tau)$

$= D_n X_n$ QED.

Note Claim 1 $\Rightarrow$ X_n is the wealth of a self-financing portfolio.

Claim 2: $X_\tau = G_\tau$

Pf:
$$X_n = \frac{1}{D_n}\tilde{E}_n\left(D_\tau G_\tau\right)$$

$$\Rightarrow X_n \cdot \mathbb{1}_{\{\tau=n\}} = \frac{1}{D_n}\left[\tilde{E}_n\left(D_\tau G_\tau\right)\right]\mathbb{1}_{\{\tau=n\}}$$

$$= \frac{1}{D_n}\tilde{E}_n\left(\mathbb{1}_{\{\tau=n\}} D_\tau G_\tau\right) \quad (\because \tau \text{ is a stopping time})$$

$$= \frac{1}{D_n}\tilde{E}_n\left(\mathbb{1}_{\{\tau=n\}} D_n G_n\right)$$

$$= \frac{1}{\cancel{D_n}}\left(\mathbb{1}_{\{\tau=n\}} \cancel{D_n} G_n\right) = \mathbb{1}_{\{\tau=n\}} G_\tau .$$

$$\Rightarrow X_\tau \mathbb{1}_{\{\tau=n\}} = G_\tau \mathbb{1}_{\{\tau=n\}} \qquad \boxed{\forall n}$$

$$\Rightarrow X_\tau = G_\tau \qquad \text{QED}$$

Hence: X is a replicating portfolio. ($\Rightarrow$ security can be replicated).

Compute AFP_G, when $n \leq \tau$:

AFP when $\tau \geq n$ $= \underbrace{X_n \mathbb{1}_{\{\tau \geq n\}}} = \mathbb{1}_{\{\tau\geq n\}} \left(\frac{1}{D_n} \tilde{E}_n \left(D_\tau G_\tau \right) \right)$

$$= \frac{1}{D_n} \tilde{E}_n \left(\underbrace{D_\tau G_\tau}\, \underbrace{\mathbb{1}_{\{\tau \geq n\}}} \right)$$

[$\because$ τ is a stopping time
$\Rightarrow \{\tau \geq n\} \in \mathcal{F}_n$]

QED.

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$\textcircled{1}\quad X_N \mathbf{1}_{\{\sigma=N\}} = G_N \mathbf{1}_{\{\sigma=N\}}$$

$$\textcircled{2}\quad X_n \mathbf{1}_{\{\sigma\geqslant n\}} = G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma>n\}} \tilde{\boldsymbol{E}}_n X_{n+1}\,.$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1)\,.$$

Pf: Know $X_n = \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma) \Rightarrow X_N \mathbb{1}_{\{\sigma=N\}} = G_\sigma \mathbb{1}_{\{\sigma=N\}}$. $\Rightarrow$ ①

Define ②:

Note: $D_n X_n = \tilde{E}_n (D_{n+1} X_{n+1})$ ($\because D_n X_n$ is a $\tilde{P}$ mg)

$\Rightarrow X_n \mathbb{1}_{\{\sigma\geq n\}} = \mathbb{1}_{\{\sigma \geq n\}} \left(\frac{1}{D_n} \tilde{E}_n (D_{n+1} X_{n+1}) \right)$

$$= \mathbb{1}_{\{\sigma = n\}}\left(\frac{1}{D_n}\tilde{E}_n\left(D_{n+1}X_{n+1}\right)\right) + \mathbb{1}_{\{\sigma > n\}}\left(\frac{1}{D_n}\tilde{E}_n\left(D_{n+1}X_{n+1}\right)\right)$$

$$= \mathbb{1}_{\{\sigma = n\}}\frac{1}{\cancel{D_n}}\left(\cancel{D_n}X_n\right) + \frac{1}{D_n}\tilde{E}_n\left(\mathbb{1}_{\{\sigma > n\}}\underline{\underline{D_{n+1}}}X_{n+1}\right)$$

$$= \mathbb{1}_{\{\sigma = n\}}G_\sigma + \frac{1}{(1+r)}\tilde{E}_n\left(\mathbb{1}_{\{\sigma > n\}}X_{n+1}\right)$$

QED

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$X_N \mathbf{1}_{\{\sigma=N\}} = G_N \mathbf{1}_{\{\sigma=N\}}$$

$$X_n \mathbf{1}_{\{\sigma\geqslant n\}} = G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma>n\}} \tilde{\boldsymbol{E}}_n X_{n+1}\,.$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1)\,.$$

Last time: $\tau \longrightarrow$ stopping time (finite)

Security pays G_τ at the random time τ.

As before, we will use state processes to find practical algorithms to price securities.

Proposition 6.43. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let $A_1, \dots, A_N \subseteq \mathbb{R}^d$, with $A_N = \mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\}.$$

Let $g_0, \dots g_N$ be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. Then *The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n) \mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation*

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r} \Big(\tilde{p} f_{n+1}(h_{n+1}(y,1)) + \tilde{q} f_{n+1}(h_{n+1}(y,-1)) \Big) \quad \leftarrow y \in \operatorname{Range}(Y_n)$$

Eg: Up rebate option → pays first time ~~when~~ $S_n \geq U$.

$d=2$ → Set $Y_n = (S_n, M_n)$ $\quad (M_n = \max\{S_0, S_1, S_2 \dots S_n\})$

$A_n = \mathbb{R} \times [U, \infty) = \{(s,m) \mid m \geq U\}$. $\quad (Y_n \in A_n \Rightarrow M_n \geq U \Rightarrow$ sec pays$)$

Proof: ① At time N: $AFP = X_N \mathbb{1}_{\{\tau = N\}} = G_N \mathbb{1}_{\{\tau = N\}}$.

$$= \underline{\underline{g_N(Y_N)}} \mathbb{1}_{\{\tau = N\}}$$

$\Rightarrow f_N(y) = g_N(y)$.

② Ind step: Compute f_n:

$$AFP \text{ at time } n = X_n \mathbb{1}_{\{\tau > n\}} = G_n \underline{\mathbb{1}_{\{\tau = n\}}} + \left(\tilde{E}_n \mathbb{1}_{\{\tau > n\}} X_{n+1} \right) \frac{1}{1+r}$$

$$= g_n(Y_n) \mathbb{1}_{\{Y_n \in A_n\}} + \tilde{E}_n \mathbb{1}_{\{Y_n \notin A_n\}} f_{n+1}(Y_{n+1}) \frac{1}{1+r}$$

$$Y_{n+1} = h_{n+1}(Y_n, \omega_{n+1})$$

$$= g_n(Y_n)\,\mathbb{1}_{\{Y_n \in A_n\}} + \frac{1}{1+r}\,\tilde{E}_n\,\mathbb{1}_{\{Y_n \notin A_n\}}\, f_{n+1}\big(h_{n+1}(\underline{Y_n}, \underline{\omega_{n+1}})\big)$$

indep lemma

$$= g_n(Y_n)\,\mathbb{1}_{\{Y_n \in A_n\}} + \frac{1}{1+r}\,\mathbb{1}_{\{Y_n \notin A_n\}}\Big(\tilde{p}\, f_{n+1}\big(h_{n+1}(Y_n, 1)\big) + \tilde{q}\, f_{n+1}\big(h_{n+1}(Y_n, -1)\big)\Big).$$

fn of Y_n. Set $Y_n = y$ & get the rel.

QED.

6.4. $\boxed{\textbf{Optional Sampling.}}$ Consider a market with a few risky assets and a bank. (int rate r)

Question 6.44. *If there is no arbitrage opportunity at time N, can there be arbitrage opportunities at time $n \leqslant N$? How about at finite stopping times?*

$X_0 = 0$, X = wealth of a self fin portfolio.

$$X_N \geq 0 \implies X_N = 0 \text{ a.s.}$$

Can $\exists\, n \leq N$ + $X_0 = 0$, $X_n \geq 0 \overset{?}{\implies} X_n = 0$

Yes: Must have $X_n = 0$. (Otherwise → transfer all \$ to bank at time n & get an arb opportunity at time N).

Can $\exists$ a finite stopping time τ $\ni$ $X_0 = 0$, $X_{\tau} \geq 0$ & $X_\tau \neq 0$

identically?

Claim NO. If such a τ exists,

transfer wealth to bank at time τ

& get an arb opp at time N. (some details to be checked)

Proposition 6.45. *There is no arbitrage opportunity at time N if and only if there is no arbitrage opportunity at any finite stopping time.*

(You check)

Question 6.46. *Say M is a martingale. We know $\boldsymbol{E}M_n = \boldsymbol{E}M_0$ for all n. Is this also true for stopping times?*

$$\left(\because \quad EM_{n+1} = E E_n M_{n+1} = E M_n \right)$$

Is $EM_\tau = EM_0$ for finite stopping times τ?

$$X_n \begin{cases} +1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}, \quad X_n \text{ iid.}$$

$\tau =$ first time $M_n = \underline{1000}$

$$\boxed{\begin{array}{l} M_n = \sum_1^n X_k \\ Q: EM_n = 0. \\ \quad M \text{ is a mg.} \\ Q2: \underline{EM_\tau} = \underline{1000} \end{array}}$$

Theorem 6.47 (Doob's optional sampling theorem). *Let τ be a $\boxed{\text{bounded stopping}}$ time and M be a martingale. Then* $\boxed{\boldsymbol{E}_n M_\tau} = M_{\tau \wedge n}$.

Note OST $\Rightarrow$ $E M_\tau = E_0 M_\tau \overset{OST}{=} M_{\tau \wedge 0} = M_0$

OST $\Rightarrow$ $E M_\tau = M_0$ (not random)

$= E M_0.$

ne

$Y = (Y^1 \dots Y^d)$ $\quad d \in \{1, 2, \dots\}$

State process : Security pays $\underline{g_N(Y_N)}$ at time N.

(1) $\Longrightarrow$ AFP at time n $= \underline{g_n(Y_n)}$ for some fn g_n.

Thm (1): $Y_{n+1} = h_{n+1}(Y_n, \underline{\omega_{n+1}}) \Longrightarrow Y$ is a state process.

Thm (2) Y is markov (& int rate not random) $\Rightarrow$

Question 6.46. *Say M is a martingale. We know $\boldsymbol{E}M_n = \boldsymbol{E}M_0$ for all n. Is this also true for stopping times?*

$$\boldsymbol{E}M_0 = M_0$$

Q: $E M_\tau$ for some stopping time τ

$$\| $$

$$E M_0$$

Theorem 6.47 ($\boxed{\text{Doob's optional sampling theorem}}$). *Let τ be a bounded stopping time and M be a martingale. Then $\boldsymbol{E}_n M_\tau = M_{\tau\wedge n}$.*

OST.

Remark 1: Know $E_n M_{n+1} = M_n$.

Choose $\tau = n+1$ (stopping time)

OST: $E_n M_\tau = M_{\tau \wedge n}$

$E_n M_{n+1}$

$M_{n \wedge (n+1)} = M_n$. ✓

Proof of OST: τ is bdd $(\Rightarrow P(\tau=\infty)=0)$

$$E_n M_\tau = E_n\left(\sum_{k=0}^{N} \mathbb{1}_{\{\tau=k\}} M_k\right) = \sum_{k=0}^{N} E_n \mathbb{1}_{\{\tau=k\}} M_k$$

$$= \sum_{k=0}^{n} E_n \mathbb{1}_{\{\tau=k\}} M_k + \sum_{k=n+1}^{N} E_n \mathbb{1}_{\{\tau=k\}} M_k$$

can remove ($\because \tau$ a stopping time $\Rightarrow \{\tau=k\} \in \mathcal{F}_k \ \forall k$)

$$= \sum_{k=0}^{n} \mathbb{1}_{\{\tau=k\}} M_k + \sum_{k=n+1}^{N} E_n \mathbb{1}_{\{\tau=k\}} E_k M_N$$

($\because M$ is a mg)

$$= \sum_{k=0}^{n} 1_{\{\tau=k\}} M_k + \sum_{k=n+1}^{N} \underbrace{E_n E_k}_{E_n} \left(1_{\{\tau=k\}} M_N\right)$$

$$= \quad '' \quad + \sum_{k=n+1}^{N} E_n \left(1_{\{\tau=k\}} M_N\right)$$

$$= \quad '' \quad + E_n \left(\underbrace{\left(\sum_{k=n+1}^{N} 1_{\{\tau=k\}} \right)} M_N \right)$$

$$= \quad '' \quad + E_n \left(1_{\{\tau>n\}} M_N\right)$$

$\left(\because \{\tau>n\} = \{\tau \le n\}^c \in \mathcal{F}_n \right)$

$$= \sum_{k=0}^{n} \mathbf{1}_{\{\tau=k\}} M_{\cancel{k}\,\tau} + \mathbf{1}_{\{\tau>n\}} \underbrace{E_n M_N}$$

$$= \mathbf{1}_{\{\tau\le n\}} M_{\tau\wedge n} + \mathbf{1}_{\{\tau>n\}} M_{n\wedge\tau}$$

$$= \left(\mathbf{1}_{\{\tau\le n\}} + \mathbf{1}_{\{\tau>n\}}\right) M_{n\wedge\tau} = M_{n\wedge\tau}$$

QED.

Proposition 6.48. *Suppose a market admits a risk neutral measure. If X is the wealth of a self-financing portfolio and τ is a finite stopping time such that $\boxed{X_0 = 0,}$ and $X_\tau \geqslant 0$, then $X_\tau = 0$.* (a.s)

Remark 6.49. This is simply an alternate proof of Proposition 6.45.

$\tilde{E}(D_n X_n)$ is a mg whenever X_n is the wealth of a self financing Pf.

Know $D_n X_n$ is a mg under $\tilde{P}$ (RNM).

$$\longrightarrow E(D_\tau X_\tau) = E_0(D_\tau X_\tau) \overset{\text{OST}}{=} D_0 X_0 = 0$$

Note $X_\tau \geq 0$ a.s. $\Rightarrow$ $D_\tau X_\tau \geq 0$ a.s. $(\because D_\tau > 0)$

Since $D_\tau X_\tau \geq 0$ & $E D_\tau X_\tau = 0$ $\Longrightarrow$ $D_\tau X_\tau = 0$ (a.s.)

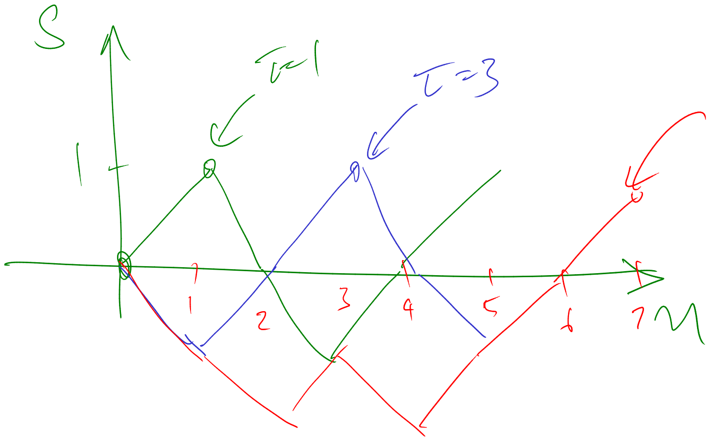
$\Rightarrow X_\tau = 0$ a.s. QED.

Remark: OST $\Rightarrow$ If M is a mg & τ is a $\boxed{\text{bdd}}$ stopping time $\Big\} \Rightarrow \mathbb{E} M_\tau = \underline{\mathbb{E}} M_0$

$$\left(\because \mathbb{E} M_\tau = \mathbb{E}\, \underline{\mathbb{E}_0} M_\tau \overset{\text{OST}}{=\!=} \mathbb{E} M_0 \right).$$

Question 6.50 (Gamblers ruin). *Suppose $N = \infty$. Let X_n be i.i.d. random variables with mean 0, and let $S_n = \sum_1^n X_k$. Let $\tau = \min\{n \mid S_n = 1\}$. (It is known that $\tau < \infty$ almost surely.) What is $\boldsymbol{E} S_\tau$? What is $\lim_{N\to\infty} \boldsymbol{E} S_{\tau\wedge N}$?* $= 0$

$S_0 = 0, \quad S_1 = X_1, \quad S_2 = X_1 + X_2 \quad \cdots$



Note ① τ is a stopping time

② S_n is a mg

$$\left(\because E_n S_{n+1} = E_n(S_n + X_{n+1}) = S_n + E X_{n+1} = S_n\right)$$

③ $E S_\tau = E\,1 \neq \underbrace{E S_n}_{0} \ \forall n$

(4) Claim: $\tau < \infty$ almost surely (τ is a finite stopping time)

(5) Does this contradict OST (No because τ is NOT bdd)

Does this give you an arb op at gambling? $\leftarrow$ NO

(because you can run out of money before winning \$1)

6.5. American Options. An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.51. Let $G_0, G_1, \ldots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.52. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.53. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*

Strategy I: Let σ be a finite stopping time, and consider an option with (random) maturity time σ and payoff G_σ. Let V_0^σ denote the arbitrage free price of this option. The arbitrage free price of the American option *should be* $V_0 = \max_\sigma V_0^\sigma$, where the maximum is taken over all *finite* stopping times σ.

$\to V_0^{\sigma^*} = \max_\sigma = V_0^\sigma$

Definition 6.54. The *optimal exercise time* is a stopping time σ^* that maximizes $V_0^{\sigma^*}$ over all finite stopping times.

Definition 6.55. An optimal exercise time σ^* is called *minimal* if for every optimal exercise time τ^* we have $\sigma^* \leqslant \tau^*$.

Remark 6.56. The optimal exercise time need not be unique. (The *minimal* optimal exercise time is certainly unique.)

Say have 1 american option.

Pick $\underline{\sigma} \to$ finite stopping time & sell this option with fixed maturity $\boxed{\sigma}$

Know how to price V_0^σ = price.

Sell to highest bidder $\longrightarrow$ can sell for $\max_\sigma V_0^\sigma$

max over all finite stopping times τ

Guess: Price of american option $= \underline{\max_{\tau} V_0^{\tau}} = V_0^{\tau^*}$

$\Vert$

V_0

Question 6.57. *Does this replicate an American option? Say σ^* is the optimal exercise time, and we create a replicating portfolio (with wealth process X) for the option with payoff G_{σ^*} at time σ^*. Suppose an investor cashes out the American option at time τ. Can we pay him?*

$$V_0^{\sigma^*} = \max_{\sigma} V_0^{\sigma}$$

X_n = wealth at time n of the portfolio required to replicate the option with fixed maturity σ^* & payoff G_{σ^*}.

Q: Does this "replicate" the American option?

(No → If he exercises at a different stopping time τ, not guaranteed we can pay him)!

Strategy II: Replication. Suppose we have sold an American option with intrinsic value G to an investor. Using that, we hedge our position by investing in the market/bank, and let X_n be the our wealth at time n.

(1) Need $X_\sigma \geqslant G_\sigma$ for all finite stopping times σ. (Or equivalently $X_n \geqslant G_n$ for all n.)

(2) For (at-least) *one* stopping time σ^*, need $X_{\sigma^*} = G_{\sigma^*}$.

The arbitrage free price of this option is X_0.

Say the american option can be traded for $X_0 + \varepsilon$ at time 0 $(\varepsilon > 0)$

↳ at time 0, sell option for $X_0 + \varepsilon$
buy Rep strat for X_0, ε in bank → Wealth at any time n: $\varepsilon(1+r)^n + X_n - G_n$

or at any stopping time τ, Wealth $= \underbrace{\varepsilon(1+r)^\tau}_{\text{profit}} + \underbrace{X_\tau - G_\tau}_{\geqslant 0 \text{ (no risk)}}$ ⇒ arbitrage

Say we can trade at the american option at price $X_0 - \varepsilon$ at time 0

↳ Then at time $0 \to$ buy option for $X_0 - \varepsilon$

short rep portfolio for X_0

ε in bank.

My Wealth at time $n = G_n - X_n + \varepsilon(1+r)^n$

Wealth at any stopping time $\tau = G_\tau - X_\tau + \varepsilon(1+r)^\tau$

Choose $\tau = \tau^* =$ optimal exercise time $\Rightarrow$ Wealth $= \underbrace{G_{\tau^*} - X_{\tau^*}}_{0} + \underbrace{\varepsilon(1+r)^\tau}_{\text{profit.}} \Rightarrow$ arb!

Proposition 6.58. *In the binomial model with* $\boxed{0 < d < 1 + r < u,}$ *we must have* $\boxed{X_0 = \max\{V_0^\sigma \mid \sigma \text{ is a finite stopping time }\}}$.

Remark 6.59. The above is true in any complete, arbitrage free market.

X_n = wealth at time n of R. Portfolio above.

Pf: ① Check $X_0 \geq V_0^\tau$ $\forall$ finite stopping times τ.

Pf: For fixed exercise time τ,

AFP at time n $= \frac{1}{D_n} \tilde{E}_n (D_\tau G_\tau) \mathbb{1}_{\{n \leq \tau\}}$ (when $n \leq \tau$)

Put $n=0$: $V_0^\tau = \tilde{E}(D_\tau G_\tau) \leq \tilde{E}(D_\tau X_\tau) \overset{OST}{=} \tilde{E}(D_0 X_0)$

① R. Portfolio

$= X_0$

mg under $\tilde{P}$

$$\Rightarrow X_0 \geq V_0^{\tau} \quad \forall \tau$$

$$\Rightarrow X_0 \geq \max \{ V_0^{\tau} \mid \tau \text{ is any finite stopping time} \}.$$

② Claim: $X_0 \leq \max \{ V_0^{\tau} \mid \tau \text{ is any finite stopping time} \}$.

Pf: Choose $\tau = \tau^* =$ optimal exercise time.

Know $X_{\tau^*} = G_{\tau^*}$.

& $V_0^{\tau^*} = \tilde{E}(D_{\tau^*} G_{\tau^*}) = \tilde{E}(D_{\tau^*} X_{\tau^*}) \overset{OST}{=} \tilde{E} D_0 X_0 = X_0$

QED.

Question 6.60. *Is the wealth of the replicating portfolio (for an American option) uniquely determined?*

Last time: American options $\rightarrow$ intrinsic value $G = (G_0, G_1, \dots G_N)$

$\hookrightarrow$ Can exercise at any stopping time

① V_0^τ = AFP of an option with maturity time τ & payoff G^τ (at time 0)

(τ = finite stopping time)

② Sell to highest bidder $\Rightarrow$ AFP of American opt $= \max\limits_{\tau} V_0^\tau$

③ Replication: To replicate an american option we need to find a self fin portfolio (wealth X_n) such that

(1) $X_n \geq G_n \quad \forall n \qquad \left(\Leftrightarrow X_\tau \geq G_\tau \ \forall \text{ finite stopping times } \tau\right)$

& (2) $X_{\tau^*} = G_{\tau^*}$ for at least one finite stopping time

called ~~the~~ a optimal exercise time

last time: $X_0 = V_0^{\tau^*} = \max_\tau V_0^\tau$

Question 6.60. *Is the wealth of the replicating portfolio (for an American option) uniquely determined?*

Say X & Y are the wealth processes of ~~the~~ R. port of American option.

Know: ① $X_n \geq G_n$ & $Y_n \geq G_n$.

& ② $X_{\tau^*} = G_{\tau^*}$ for any finite stopping time τ^* $Y_{\tau^*} = G_{\tau^*}$ for 1 finite stopping time τ^*

Claim: In an AF market, $X_{\tau^*} = Y_{\tau^*}$ (& $X_{\tau^*} = Y_{\tau^*}$)

Pf: Know $X_{\tau^*} \geq G_{\tau^*} = Y_{\tau^*}$.

Also, AFP at time $0 = \tilde{E}\left(D_{\tau^*} G_{\tau^*}\right) = V_0 = Y_0$

$V_0 = X_0 \overset{OST}{=} \tilde{E}\left(D_{\tau^*} X_{\tau^*}\right) \quad \tilde{E}\left(D_{\tau^*} G_{\tau^*}\right)$

Note $D_{\tau^*} X_{\tau^*} \geq D_{\tau^*} G_{\tau^*}$

$\Rightarrow X_{\tau^*} = G_{\tau^*}$.

QED.

Claim (IOW): Before the optimal exercise time

$X_n = Y_n$

If $EZ = EW$

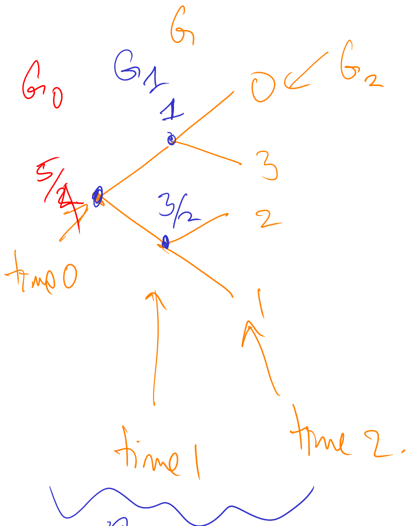
$\overset{?}{\Rightarrow} Z = W$

But

$Z \geq W$ & $EZ = EW$

$\Rightarrow Z = W$

Question 6.61. *How do you find the minimal optimal exercise time, and the arbitrage free price? Let's take a simple example first.*

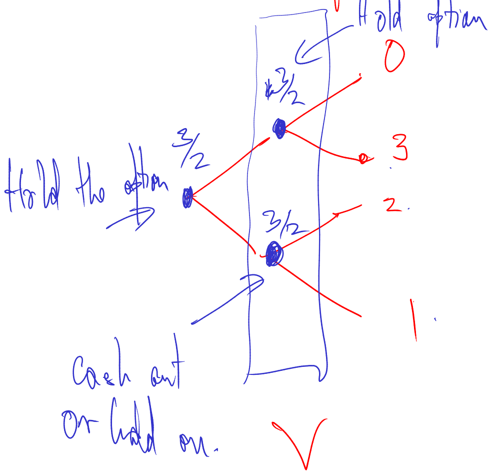


made up #'s for ex

Choose for simplicity $r = 0$, $\tilde{p} = \tilde{q} = \frac{1}{2}$.

AFP of this option is

$$\frac{\text{AFP at time } 2}{\text{AFP at time } 1}$$



Theorem 6.62. *Consider an American option with intrinsic value G. Define*

$G = (G_0, G_1, \dots, G_N)$

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time.

Remark 6.63. For the binomial model with $0 < d < 1 + r < u$ the above simplifies to

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \dots, \omega_n)\,.$$

To prove Thm 6.62: IOU $\longrightarrow$ ~~Need~~

① Need to find a self financing portfolio such with wealth process X_n such that: ⓐ $X_n \geqslant G_n \quad \forall n$ — moreover ⓒ whenever $n < \sigma^*$, $X_n = V_n > G_n$

ⓑ $X_{\sigma^*} = G_{\sigma^*}$

Last time: American option $\rightarrow$ intrinsic value $G_\cdot \rightarrow (G_0, G_1, \cdots G_N)$

↳ Can exercise at any time (finite stopping time)

Payoff at time τ is G_τ

Q: Price $\rightarrow$ $\max\{V_0^\tau \mid \tau \text{ any finite stopping time}\}$

& V_0^τ = AFP at time 0 of the security with payoff G_τ at time $\underline{\tau}$.

Replication: ① Need a self fin portfolio with wealth X &

ⓐ $X_n \geq G_n$

ⓑ $X_{\tau^*} = G_{\tau^*}$ for one stopping time τ^*.

Theorem 6.62. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

Then V_n is the arbitrage free price, at time n *and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.*

Remark 6.63. The above is true in any complete, arbitrage free market.

Remark 6.64. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega',1) + \tilde{q}V_{n+1}(\omega',-1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \dots, \omega_n)\,.$$

Proof: V_n = AFP at time n (time)

IOU: σ^* = minimal optimal exercise time

IOU: Replication.

Obs 1: The process $\boxed{D_n V_n}$ is a super-martingale under $\tilde{P}$

M_n is a martingale (under $\tilde{P}$) if $\tilde{E}_n M_{n+1} = M_n$

Def: (a) We say an adapted process M_n is a super-mg (under $\tilde{P}$) if $\tilde{E}_n M_{n+1} \leq M_n$.

(b) M is a sub-mg if $\tilde{E}_n M_{n+1} \geq M_n$.

Pf of Obs 1: NTS $D_n V_n$ is a super mg under $\tilde{P}$.

i.e. NTS $\tilde{E}_n (D_{n+1} V_{n+1}) \leq D_n V_n$

$\Leftrightarrow$ NTS $\quad V_n \geq \dfrac{\tilde{E}_n(D_{n+1} V_{n+1})}{D_n}$.

This is true because $V_n \overset{\text{def}}{=} \max\left\{G_n, \frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1})\right\}$.

$\Rightarrow D_n V_n$ is a super mg under $\tilde{P}$.

Thm: (Doob) $\underbrace{\text{(IOU Pf)}}$. Any super mg can be written as a martingale — (a predictable inc process).

Obs 2: Apply Doob to $D_n V_n$.

$\Rightarrow \exists$ a predictable $\boxed{\text{inc}}$ process A & a mg M $\ni$

$$D_n V_n = M_n - A_n \qquad (\text{Choose } \boxed{A_0 = 0})$$

Set $X_n = \dfrac{M_n}{D_n} \quad \Rightarrow \quad D_n V_n = D_n X_n - \underbrace{A_n}$

(On HW today $\rightarrow$ Explicit formula for A)

③ $\Rightarrow$ X_n = wealth of some self fin portfolio

Claim: ① $X_n \geq G_n$ & ③ On $\{n < \tau^*\}$, $X_n = V_n > G_n$

② $X_{\tau^*} = G_{\tau^*}$

Note ① + ② $\Rightarrow$ X is the wealth of the rep port of american option.
(& self fin) (& τ^* is an optimal exercise time).

③ $\Rightarrow$ τ^* is the minimal optimal exercise time.

($\because$ Say τ is any stopping time & $\tau \leq \tau^*$
& $P(\tau < \tau^*) > 0$

Then $\tilde{E}(D_\tau X_\tau)$ $E(D^*_\tau G^*_\tau)$

will return to this shortly.

↳ Pf of claim: ① $X_n \geq G_n$.

Note $D_n V_n = D_n X_n - A_n \Leftrightarrow X_n = V_n + \frac{A_n}{D_n}$

Note A_n is increasing & $A_0 = 0 \Rightarrow A_n \geq 0$

$\Rightarrow X_n \geq V_n = \max\{G_n, \_\_\_\_\_\}$

$\Rightarrow X_n \geq G_n$. QED.

Claim ②: $X_{\tau^*} = G_{\tau^*}$

$\tau^* = \min\{n \mid V_n = G_n\}$.

$\boxed{\underline{\text{Claim}}: \ \forall n \le \tau^*, \text{ we must have } A_n = 0}$

$(\Rightarrow X_n = V_n \text{ whenever } n \le \tau^*)$

$\Rightarrow X_{\tau^*} = V_{\tau^*} = G_{\tau^*}$.

IOU.

$\hookrightarrow$ Let's prove this: note $V_n = \max\left\{ \frac{1}{D_n} \tilde{E}_n(D_{n+1} V_{n+1}), G_n \right\}$,

$\Rightarrow$ whenever $n \le \tau^*$, we must have $V_n = \frac{1}{D_n} \tilde{E}_n(D_{n+1} V_{n+1})$

$\Rightarrow$ when $n \lneq \tau^*$, $D_n V_n = \tilde{E}_n(D_{n+1} V_{n+1})$

Note $D_n X_n = \tilde{E}_n(D_{n+1} X_{n+1}) \quad \forall n.$

& $D_n V_n = D_n X_n \,\underline{+}\, A_n$

$$\mathbb{1}_{\{n \lneq \tau^*\}} \tilde{E}_n(D_{n+1} V_{n+1}) = \tilde{E}_n\left(\mathbb{1}_{\{n \lneq \tau^*\}} D_{n+1} V_{n+1}\right) = \mathbb{1}_{\{n \lneq \tau^*\}} D_n V_n.$$

$\|$

$$\mathbb{1}_{\{n \lneq \tau^*\}} \tilde{E}_n\Big(D_{n+1} X_{n+1} - \underbrace{A_{n+1}}\Big)$$

$$(\because A \text{ is pred})$$

$$\|$$

$$\mathbb{1}_{\{n \lneq \tau^*\}}\Big(D_n X_n - A_{n+1}\Big) = \mathbb{1}_{\{n \lneq \tau^*\}} D_n V_n$$

$$\Rightarrow \mathbb{1}_{\{n \lneq \tau^*\}}\Big(\underbrace{D_n X_n - A_n}_{D_n V_n} + A_n - A_{n+1}\Big) = \mathbb{1}_{\{n \lneq \tau^*\}} D_n V_n$$

$$\Rightarrow \mathbb{1}_{\{n < \tau^*\}} (A_{n+1} - A_n) = 0$$

$$\boxed{\Rightarrow A_n = 0 \quad \forall n \leq \tau^*.}$$

(Will finish the rest next time)

Theorem 6.62. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.

Remark 6.63. The above is true in any complete, arbitrage free market.

Remark 6.64. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \dots, \omega_n)\,.$$

Last time: Proved this option can be replicated.

IOU: σ^* is the minimal optimal exercise time

IOU: V_n is the AFP.

Theorem 6.65. *Consider the ~~Binomial model~~ with* $0 < d < 1 + r < u$, *and a state process* $Y = (Y^1, \dots, Y^d)$ *such that* $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega'), \omega_{n+1})$, *where* $\omega' = (\omega_1, \dots, \omega_n)$, $\omega = (\omega', \omega_{n+1}, \dots, \omega_N)$, *and* $h_0, h_1, \dots, h_N$ *are* N *deterministic functions. Let* $g_0, \dots, g_N$ *be* N *deterministic functions, let* $G_k = g_k(Y_k)$, *and consider an American option with intrinsic value* $G = (G_0, G_1, \dots, G_N)$. *The* pre-exercise price *of the option at time* n *is* $f_n(Y_n)$, *where*

$$f_N(y) = g_N(y) \quad \text{for } y \in \operatorname{Range}(Y_N), \qquad f_n(y) = \max\Big\{ g_n(y), \frac{1}{1+r}\Big(\tilde p f_{n+1}(h_{n+1}(y, \overset{1}{u})) + \tilde q f_{n+1}(h_{n+1}(y, \overset{-1}{\cancel{d}}))\Big)\Big\}, \quad \text{for } y \in \operatorname{Range}(Y_n).$$

The minimal optimal exercise time is $\sigma^* = \min\{n \mid f_n(Y_n) = g_n(Y_n)\}$.

Pf: Know AFP at time $n = V_n$, where

$$V_N = G_N, \qquad V_n = \max\Big\{ \frac{1}{D_n} \tilde E_n D_{n+1} V_{n+1}, G_n \Big\}$$

$$= g_N(Y_N) = f_N(Y_N) \qquad = \max\Big\{ \frac{1}{1+r} \tilde E_n f_{n+1}(Y_{n+1}), G_n \Big\}$$

(Assume $V_{n+1} = f_{n+1}(Y_{n+1})$)

$$= \max\left\{ g_n(Y_n),\ \frac{1}{1+r}\tilde{E}_n f_{n+1}\left(h_{n+1}(Y_n, \omega_{n+1}) \right) \right\}$$

indep lem.

$$= \max\left\{ g_n(Y_n),\ \frac{1}{1+r}\left(f_{n+1}(h_{n+1}(Y_n,+1))\tilde{p} + f_{n+1}(h_{n+1}(Y_n,-1))\tilde{q} \right) \right\}$$

Set $y = Y_n \Rightarrow$ done! QED.

m

Pay IOU's: $V_N = G_N$. $V_n = \max\left\{ G_n , \frac{1}{D_n} \tilde{E}_n (D_{n+1} V_{n+1}) \right\}$

$$\tau^* = \min\{ n \mid V_n = G_n \}.$$

IOU: $V_n = AFP$ (maybe next time)

τ^* = minimal optimal exercise time. ← Prove. (Hopefully).

Last time: American option can be replicated. ← Review (+ fix typo)

Main Idea: Doob decomposition: Super mg = Mg − (predictable & inc).

Super mg : $\mathbb{E}_n Y_{n+1} \le Y_n$

Note : $V_n = \max\left(G_n, \frac{1}{D_n}\tilde{\mathbb{E}}_n(D_{n+1}V_{n+1})\right) \geq \frac{1}{D_n}\tilde{\mathbb{E}}_n(D_{n+1}V_{n+1})$

$\Rightarrow D_n V_n \geq \tilde{\mathbb{E}}_n(D_{n+1}V_{n+1}) \Rightarrow D_n V_n$ is a $\tilde{P}$ super mg.

2 Doob $\Rightarrow D_n V_n = M_n - A_n$: $M_n \to \tilde{P}$ ~~super~~ mg

$A_n \to$ Predictable inc. & $A_0 = 0$

Write $M_n = D_n X_n$.

$$\boxed{\Rightarrow D_n V_n = D_n X_n - A_n}$$

③ Last time X_n = wealth of a self fin portfolio

$(X_n \geq G_n \ \ldots)$

④ Claim $A_{\tau^*} = 0$ (A increasing & $A_0 = 0$

$\Rightarrow A_n = 0 \quad \forall n \leq \tau^*$)

Pf: $$\mathbb{1}_{\{n < \tau^*\}} V_n = \frac{1}{D_n} \tilde{E}_n (V_{n+1} D_{n+1}) \mathbb{1}_{\{n < \tau^*\}}$$

(by def of V_n & τ^*).

$$\Rightarrow \quad 1_{\{n<\tau^*\}}(D_n V_n) = 1_{\{n<\tau^*\}} \tilde{E}_n(D_{n+1} V_{n+1})$$

Also note $\quad \tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n \quad (\because D_n X_n \text{ is a } \tilde{P} \text{ mg})$

and $\quad \tilde{E}_n(A_{n+1}) = A_{n+1} \quad (\because A \text{ is predictable})$

Know $D_{n+1} V_{n+1} = D_{n+1} X_{n+1} - A_{n+1}$. mult by $1_{\{n<\tau^*\}}$ & take $\tilde{E}_n$

$$\Rightarrow 1_{\{n<\tau^*\}} D_n V_n = 1_{\{n<\tau^*\}} \left[(D_n X_n) - A_{n+1} \right]$$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} D_n V_n = \mathbb{1}_{\{n<\tau^*\}} \Big[\underbrace{D_n X_n - A_n}_{D_n V_n} + A_n - A_{n+1} \Big]$$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} \left(A_n - A_{n+1} \right) = 0$$

$$\left.\begin{array}{l} \Rightarrow \mathbb{1}_{\{n<\tau^*\}} A_{n+1} = \mathbb{1}_{\{n<\tau^*\}} A_n \\ \text{Also know } A_0 = 0 \end{array}\right\} \Rightarrow A_{\tau^*} = 0$$

$$\Rightarrow \mathbb{1}_{\{n \leq \tau^*\}} A_n = 0$$

$$\boxed{(\text{i.e. } A_n = 0 \ \forall\, n \leq \tau^*)}$$

Claim: τ^* is the minimal optimal exercise time.

Recall: optimal exercise time τ: $\tilde{E}(G_\tau D_\tau) = V_0 = \max_\tau \tilde{E}(G_\tau D_\tau)$

IOU (1) ~~Clearly~~ τ^* is an optimal exercise time.

IOU ② τ^* is the minimal optimal exercise time

(AFP)

Last time: American option intrinsic value G_n.

Let $V_N = G_N$ & $V_n = \max\left\{ G_n , \underline{\frac{1}{D_n}\tilde{E}_n\left(D_{n+1}V_{n+1}\right)} \right\}$

Then claim: ① The option can be replicated

② AFP at time n is V_n ⟵

& ③ Minimal optimal exercise time is $\sigma^* = \min\{ n \mid V_n = G_n \}$

Pf: Last time: ① $D_n V_n \geq \tilde{E}_n\left(D_{n+1} V_{n+1}\right)$ ($\because D_n V_n$ is a $\tilde{P}$ super mg)

(2) Doob decomposition: $D_n V_n = \underbrace{P_n X_n}_{\tilde{P}\text{ mg}} - \underbrace{A_n}_{}$

(IOU)

predictable, inc, & $A_0 = 0$

(3) Last time: Checked $A_{\tau^*} = 0$ $\left(\Rightarrow \mathbb{1}_{\{n \leq \tau^*\}} A_n = 0\right)$.

(4) Claim $X = \text{wealth}_n$ of the rep portfolio (proves $\Rightarrow$ Option can be replicated).

(already did this. Review:

NTS. (a) $X_n \geq G_n$ ✓

& (b) For one finite stopping time $X_{\tau^*} = G_{\tau^*}$ ✓

Check (a): $D_n X_n = D_n V_n + A_n \Rightarrow X_n = \underbrace{V_n}_{} + \underbrace{\frac{A_n}{D_n}}_{\geq 0}$

$V_n \geq G_n$ (by formula)

$\Rightarrow X_n \geq V_n \geq G_n$.

(b) At time τ^*, claim $X_{\tau^*} = V_{\tau^*} = G_{\tau^*}$

(Pf: By def of τ^*, $V_{\tau^*} = G_{\tau^*}$. & know $A_{\tau^*} = 0$).

⑤ Check $\sigma^* =$ minimal optimal exercise time.

τ^* is an optimal exercise time if $\tilde{E}(D_{\tau^*} G_{\tau^*}) = \max_{\tau} \tilde{E}(D_\tau G_\tau)$

(max over all finite stopping times τ)

Note $\tilde{E}(D_\tau G_\tau) \leq \tilde{E}(D_\tau X_\tau) \overset{OST}{=} D_0 X_0$ ($\because D_n X_n$ is a $\tilde{P}$ mg)

$\overset{OST}{=} \tilde{E}(D_{\sigma^*} X_{\sigma^*}) = \tilde{E}(D_{\sigma^*} G_{\sigma^*})$

$\Rightarrow$ σ^* is optimal!

Check τ^* is the <u>minimal</u> optimal exercise time.

Say τ^* is any optimal exercise time

$$\Rightarrow \tilde{E}(D_{\tau^*} G_{\tau^*}) = \max_{\tau} \tilde{E}(D_\tau G_\tau) = \tilde{E}(D_{\tau^*} G_{\tau^*})$$

($\because \tau^*$ is optimal).

$$= \tilde{E}(D_{\tau^*} X_{\tau^*}) \overset{OST}{\leq} D_0 X_0 \overset{OST}{=} \tilde{E}(D_{\tau^*} X_{\tau^*})$$

$$\Rightarrow \tilde{E}(D_{\tau^*} G_{\tau^*}) = \tilde{E}(D_{\tau^*} X_{\tau^*})$$

$$\Rightarrow \quad D_{\tau^*} G_{\tau^*} = \cancel{X}\, D_{\tau^*} X_{\tau^*} \qquad \left(\because \; G_{\tau^*} \leq X_{\tau^*} \text{ from above!} \right)$$

$$\Rightarrow \quad G_{\tau^*} = X_{\tau^*}$$

$$\Rightarrow \quad V_{\tau^*} = G_{\tau^*}$$

$$\Rightarrow \quad \nu^* \leq \tau^* \qquad \left(\because \; \nu^* = \text{first time } V_n = G_n \right)$$

$\Rightarrow$ ν^* is the <u>minimal</u> optimal exercise time.

Let's also check V_n = <u>AFP</u> at time n.

Case I : Say at time n we Buy one American option at price V_n.

Short cash/stock for $-V_n$

at time n $\begin{cases} \to 1 \text{ option } (+) \\ \to -V_n \text{ in cash/stock.} \end{cases}$

Say we chose to first trade this option at time τ. ($\tau \geq n$)

Say our cash/stock portfolio is worth Y_k at time k.

(Note $Y \to$ wealth of a self financing portfolio).

At time τ, wealth $= V_\tau - Y_\tau$

Note $\tilde{E}_n(D_\tau(V_\tau - Y_\tau)) = \tilde{E}_n(D_\tau V_\tau) - \underbrace{\tilde{E}_n(D_\tau Y_\tau)}$ (OST since DY is a $\tilde{P}$ mg)

OST for super mg's (IOU) $\longrightarrow$ (OST)

$$\leq D_n V_n - D_n Y_n$$

$$= D_n V_n - D_n V_n = 0$$

$\Rightarrow \tilde{E}_n(D_\tau(V_\tau - Y_\tau)) \leq 0 \Rightarrow$ $\boxed{\text{no arbitrage opportunity}}$

Case II: Say at time n we sell ~~one~~ 1 options.

Wealth at time n $\begin{cases} V_n \text{ in stock / cash.} \\ \text{short 1 option (worth } V_n) \end{cases}$

Y_n = wealth of my portfolio in stock / cash.

Client cashes at the option when it is optimal for them

Say the client ~~ca~~ sells this option at time $\tau^* \vee n = \max\{\tau^*, n\}$.

Claim: No arb opportunity at time σ^*.

$(*)$ ... but $\left(D_{\sigma^* \vee n} V_{\sigma^* \vee n} - D_n V_n\right) = \left(D_{\sigma^* \vee n} X_{\sigma^* \vee n} - D_n X_n\right)$

$$- \underbrace{\left(A_{\sigma^* \vee n} - A_n\right)}_{0 \;\left(\because A_n = 0 \;\forall\; n \le \sigma^*\right)}$$

$\Rightarrow$ at time $\sigma^* \vee n$ we have

$$E_n\left(D_{\sigma^* \vee n} Y_{\sigma^* \vee n} - D_{\sigma^* \vee n} V_{\sigma^* \vee n}\right) \underset{(*)\,\&\,\mathrm{OST}}{=}$$

$$E_n D_{\tau^* \vee n} Y_{\tau^* \vee n} - D_n V_n$$

$$\overset{OST}{=} \cancel{\varnothing}\, D_n Y_n - D_n V_n = 0$$

$\Rightarrow$ No arbitrage opportunity.

$r \geq 0$

Theorem 6.66. *Let g be a convex function with $g(0) = 0$, and let $G_n = g(S_n)$. Consider an American option with intrinsic value $G_n = g(S_n)$. You should never exercise this option early.*

i.e. N = maturity time is an optimal exercise time.

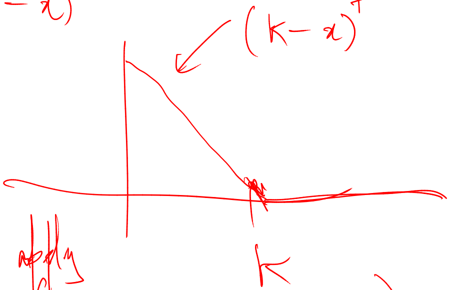
Corollary 6.67. *The arbitrage free price of an American call and European call are the same.*

American call strike K: $g(x) = (x-K)^+$

(convex & $g(0)=0$)
Thm applies

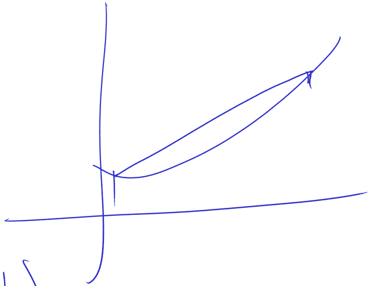
American put strike K:

$g(x) = (K-x)^+$

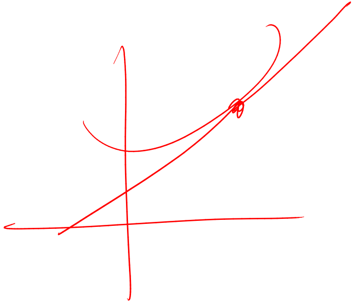


Thm does not apply
(convex, $g(0) \neq 0$).

Pf of thm: ① Convex: fn always lies below the chord (secant)



(⇔ fn always lies above the tangent)



② Recall Jensen's inequality

If g is convex & X is a RV then

$$Eg(X) \geq g(EX)$$ (on HW)

③ Conditional Jensen: $E_n\, g(X) \geq g(E_n X)$

Pf of Thm: $G_n = g(S_n)$ (intrinsic value)

$g(0) = 0$ & g convex.

AFP at time $N = g(S_N)$

AFP at time $n = V_n = \max\left\{ g(S_n), \frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1}) \right\}$

Let $n = N-1$: Claim: $V_n = \max\left\{ g(S_n), \frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1}) \right\}$

$= \max\left\{ g(S_n), \frac{1}{D_n}\tilde{E}_n(D_{n+1}\, g(S_{n+1})) \right\}$

$$= \frac{1}{D_n} \tilde{E}_n \left(D_{n+1}\, g(S_{n+1}) \right) \Big)$$

(i.e. : $g(S_n) \leq \frac{1}{D_n} \tilde{E}_n \left(D_{n+1}\, g(S_{n+1}) \right) \Big)$.

(Claim $\Rightarrow$ at time $n = N-1$, it is not in your interest to exercise. ~~cash out~~).

($\Rightarrow$ By backward induction $\Rightarrow$ at any time $\leq N$, it is not in your interest to exercise)

Pf of claim: NTS $g(S_n) \le \frac{1}{D_n}\tilde{E}_n\left(D_{n+1}\, g(S_{n+1})\right)$ ($\Rightarrow$ Claim $\Rightarrow$ thm $\Rightarrow$ QED)

Pf: $\frac{1}{D_n}\tilde{E}_n\left(D_{n+1}\, g(S_{n+1})\right) = \frac{D_{n+1}}{D_n}\tilde{E}_n\, g(S_{n+1})$

Conditional Jensen

$$\ge \frac{D_{n+1}}{D_n}\, g\left(\tilde{E}_n S_{n+1}\right)$$

$$= \frac{D_{n+1}}{D_n}\, g\left(\frac{1}{D_{n+1}}\tilde{E}_n\left(D_{n+1} S_{n+1}\right)\right)$$

$$= \frac{D_{n+1}}{D_n}\, g\left(\frac{1}{D_{n+1}} D_n S_n\right)$$

($\because D_n S_n$ is a mg!)

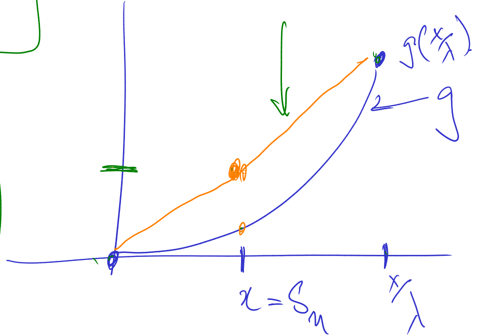
(*) $$= \frac{D_{n+1}}{D_n}\, g\left(\frac{D_n}{D_{n+1}} S_n\right) \leftarrow$$

Let $\lambda = \frac{D_{n+1}}{D_n}$.

$$\boxed{\text{Interest rate} > 0 \implies \lambda \le 1}$$

slope $= \dfrac{g(x/\lambda)}{x/\lambda}$

$$\boxed{\text{By convexity (**)} \quad g(x) \le x \frac{g(x/\lambda)}{x/\lambda} = \lambda\, g\left(\frac{x}{\lambda}\right)}$$



$\therefore$ From $(*)$ $\frac{1}{D_n}\tilde{E}_n D_{n+1}\, g(S_{n+1}) \geq \frac{D_{n+1}}{D_n}\, g\left(\frac{D_n}{D_{n+1}} S_n\right)$

$\overset{(**)}{\geq} g(S_n)$. $\Rightarrow$ Claim $\Rightarrow$ QED!

6.6. Doob Decomposition and Optimal Stopping.

Theorem 6.68 (Doob decomposition). *Any adapted process can be* $\boxed{\textit{uniquely}}$ *expressed as the sum of a martingale and a predictable process.* starting at 0

$X_n \to$ adapted process (X_n is $\mathcal{F}_n$ meas $\forall n$).

$M_n \to$ mg ($E_n M_{n+1} = M_n$)

$A_n \to$ predictable (A_{n+1} is $\mathcal{F}_n$ meas.).

Pf: X is adapted $\Rightarrow \exists!$ mg M & a predictable process $A \ni$

$X = M + A$. & $A_0 = 0$.

Pf scratch: Say $X = M + A$.

M: mg. A: pred.

(Find M & A).

$$E_n X_{n+1} = E_n M_{n+1} + E_n A_{n+1}$$

$$= M_n + A_{n+1} = \underbrace{M_n + A_n}_{X_n.} - A_n + A_{n+1}$$

$$\Rightarrow E_n X_{n+1} = X_n + A_{n+1} - A_n$$

$$\Rightarrow A_{n+1} = A_n + \left(E_n X_{n+1} - X_n\right)$$

~~Def~~ Use this to define A_{n+1} (start with $A_0 = 0$)

Set $M_{n+1} = X_{n+1} - A_{n+1}$

Should give me the desired decomposition

(do backward $\Rightarrow$ QED Next time).

Q3 Midterm:

$$\text{Game} \begin{cases} \$1 & \text{prob } 1/2 \\ -\$1 & \text{prob } 1/2. \end{cases}$$

Q: Play repeatedly.

Start with \$50.

Stop when lose all \$ or double initial stockpile.

Q1: Prob you double \$ when you stop $= \frac{1}{2}$ (OST)

Q2: Same game, start with \$1000 $\longrightarrow$

Prob (double) when you stop $= \frac{1}{2}$ (OST)

Say now

$$\text{Game} \begin{cases} +\$1 & \text{Prob } p \in (0, 1/2) \\ -\$1 & \text{Prob } 1-p. \end{cases}$$

Prob you double your fortune when you stop?

Q3 Midterm:

$$\alpha = \ln\left(\frac{1-p}{p}\right). \quad P(\text{double}) = \frac{e^{\alpha S_0} - 1}{e^{\alpha(2S_0)} - 1}$$

① $S_0 \to$ small then $P(\text{double}) \approx 1/2$.

② $S_0 \to$ large then $P(\text{double}) \approx \dfrac{e^{\alpha S_0}}{e^{2\alpha S_0}} \approx \underbrace{e^{-\alpha S_0}}_{\text{exponentially small!!}}$

6.6. **Doob Decomposition and Optimal Stopping.**

Theorem 6.68 (Doob decomposition)**.** *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process that starts at 0. That is, if X is an adapted process there exists a unique pair of process M, A such that M is a martingale, A is predictable, $A_0 = 0$ and $X = M + A$.*

$(E_n M_{n+1} = M_n$, A_{n+1} is $\mathcal{F}_n$ meas. $\iff E_n A_{n+1} = A_{n+1})$

Pf: Guess from last time: If $X_n = M_n + A_n$.

$$E_n X_{n+1} = E_n M_{n+1} + E_n A_{n+1} = M_n + A_{n+1} =$$

$$= M_n + A_n - A_n + A_{n+1}$$

$$= X_n + (A_{n+1} - A_n).$$

$$\boxed{\Rightarrow \text{Should have } A_{n+1} = A_n + E_n X_{n+1} - X_n}$$

Know $A_0 = 0$, solve for A_n.

Pf: Let $\underline{A_0 = 0}$. $A_{n+1} = \underbrace{A_n + E_n X_{n+1} - X_n}$.

& Let $M_n = X_n - A_n$.

① A is predictable ($\because A_{n+1} = \underbrace{A_n}_{\substack{\mathcal{F}_n \text{ meas} \\ (\text{ind})}} + \underbrace{E_n X_{n+1} - X_n}_{\mathcal{F}_n\text{-meas}}$)

② $A_0 = 0$. (Known)

③ NTS M is a mg.

$$E_n(M_{n+1}) = E_n(X_{n+1} - A_{n+1}) = E_n X_{n+1} - A_{n+1} \quad (\because A \text{ pred}).$$

$$= \cancel{E_n X_{n+1}} - (A_n + \cancel{E_n X_{n+1}} - X_n)$$

$$= X_n - A_n = M_n$$

QED. (Existence).

Pf of Uniqueness: Follow scratch. If $X_n = M_n + A_n$, we are forced to have $A_0 = 0$ & $A_{n+1} = A_n + (E_n X_{n+1} - X_n)$. $\longleftarrow$ solving gives A_n uniquely ($X = M + A$ gives M uniquely) QED

Definition 6.69. We say an adapted process M is a *super-martingale* if $\boldsymbol{E}_n M_{n+1} \leqslant M_n$. (life is a super mg)

Definition 6.70. We say an adapted process M is a *sub-martingale* if $\boldsymbol{E}_n M_{n+1} \geqslant M_n$.

Example 6.71. The discounted arbitrage free price of an American option is a *super-martingale* under the risk neutral measure.

When pricing American options, we knew $D_n V_n$ is a $\tilde{P}$ super mg & used Doob to write $D_n V_n = \underbrace{M_n}_{\text{mg}} - \underbrace{A_n}_{}$

predictable, $A_0 = 0$

$\boxed{\text{\& increasing}}$

Proposition 6.72. *If X is a super-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M - A$.*

Proposition 6.73. *If X is a sub-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M + A$.*

(Similar → you check)

Pf: X → super mg.

By Doob, $X = M - A$, $A_0 = 0$, M a mg & A predictable.

$$\Rightarrow E_n X_{n+1} = E_n M_{n+1} - E_n A_{n+1} = M_n - A_{n+1} \quad ①$$

Also know $E_n X_{n+1} \leq X_n = M_n - A_n$ ②

(super mg)

① & ② $\Rightarrow$ $\cancel{M_n} - A_{n+1} = E_n X_{n+1} \leq X_n = \cancel{M_n} - A_n$

$\Rightarrow A_n \leq A_{n+1}$ $\rightarrow$ A_n is increasing QED.

Corollary 6.74. *If X is a super-martingale and τ is a bounded stopping time, then* $\boldsymbol{E}_n X_\tau \leqslant X_{\tau \wedge n}$.

Corollary 6.75. *If X is a sub-martingale and τ is a bounded stopping time, then* $\boldsymbol{E}_n X_\tau \geqslant X_{\tau \wedge n}$.

Know for a mg M & a bdd stopping time τ, $E_n M_\tau \overset{OST}{=} M_{\tau \wedge n}$

Q: Sub mg?

Q: Super mg?

Pf: Say X is a super mg, τ a bdd stopping time

WTS $E_n X_\tau \leq X_{\tau \wedge n}$

Use the Doob decomposition of X: $\exists M, A$ s.t. $X = M - A$,

M — mg, $A \to$ pred & $\boxed{\text{inc}}$
& $A_0 = 0$

~~By OST~~: $E_n X_\tau = E_n (M_\tau - A_\tau)$

$\overset{OST}{=} M_{\tau\wedge n} - E_n A_\tau$

$\leq M_{\tau\wedge n} - \underbrace{E_n A_{\tau\wedge n}}$ $\left(\because \right)$

$= M_{\tau\wedge n} - A_{\tau\wedge n}$ ($\because$ A is adapted & τ is a stopping time).

$= X_{\tau\wedge n}$. QED.

Note

$A_{\tau\wedge n} \leq A_\tau$

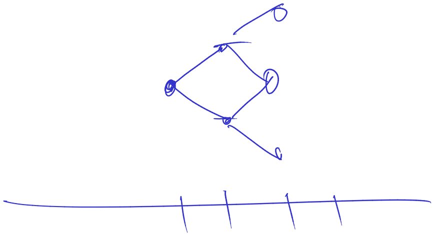
($\because$ A is inc)

Theorem 6.76 (Snell)**.** *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}\,.$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

HW Q3f:



10,000 ~ n_0

$$X_{n+1} = X_n \pm 1$$

$X_{n_0} \in \mathbb{N}$.

$\text{Range}(X_n) \approx$ even or odd ~~sense~~ integers between $\underbrace{X_{n_0} \pm (n-n_0)}$

$$f_N(x) = \{ \mathbb{1}_{\{x \geq 0\}}$$

$(x \in (-N, N)) + X_{n_0}$

$$f_n(x) = f_{n+1}\left(\underline{x+1}\right) \; \underline{\qquad}$$

$$+ \; \underline{\underline{f_{n+1}}}\left(\underline{x - 1}\right) \; \underline{\qquad}$$

Last time: Doob decomp: $X = \underbrace{M}_{\text{mg}} + \underbrace{A}_{\text{pred}}$, $A_0 = 0$

Super mg: $X = \underbrace{M}_{\text{mg}} - \underbrace{A}_{\substack{\text{pred} \\ \text{inc} \\ A_0=0}}$

$\Rightarrow$ By OST $E_n X_\tau \leq X_{\tau \wedge n}$ (X is a super mg)

(τ is bdd)

Theorem 6.76 (Snell). *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}. \qquad (n < N).$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

V is called the Snell envelope of G.

Pf: ① $V_n \geq G_n$ $\left(\because V_n = \max\{E_n V_{n+1}, G_n\}\right)$

② V is a super mg $\left(\because V_n = \max\{\ \} \geq E_n V_{n+1}\right)$

③ NTS V is the smallest super mg $\geq G$.

i.e. If W is a super mg, $W \geq G \implies W \geq V$.

Pf: Since $W_N \geq G_N = V_N \implies W_N \geq V_N$

Backward induction: Say $W_{n+1} \geq V_{n+1}$

W is a super mg $\Rightarrow$ $W_n \geq E_n W_{n+1} \geq \underline{E_n V_{n+1}}$

Also know $W_n \geq \underline{G_n}$

$$\Rightarrow W_n \geq \max\{G_n, E_n V_{n+1}\} = V_n.$$

QED.

Proposition 6.77. *If W is any martingale for which $W_n \geqslant G_n$, and for one stopping time τ^* we have $\boldsymbol{E}W_{\tau^*} = \boldsymbol{E}G_{\tau^*}$, then we must have $W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$, and $W_{\tau^* \wedge n}$ is a martingale.* (V = Snell super mg envelope of G).

Theorem 6.78. *Let $\sigma^* = \min\{n \mid V_n = G_n\}$. Then σ^* is the minimal solution to the optimal stopping problem for G. Namely, $\boldsymbol{E}G_{\sigma^*} = \max_\sigma \boldsymbol{E}G_\sigma$ where the maximum is taken over all finite stopping times σ. Moreover, if $\boldsymbol{E}G_{\tau^*} = \max_\sigma \boldsymbol{E}G_\sigma$ for any other finite stopping time τ^*, we must have $\tau^* \geqslant \sigma^*$.* (Pf is the same as we had for American options, please check).

Remark 6.79. By construction $V_{\sigma^* \wedge n}$ is a martingale.

Pf: Note $\boxed{W_n \geqslant V_n}$ $\forall n$ ($\because$ W is a mg $\Rightarrow$ W is a super mg & given $W \geqslant G \Rightarrow W \geqslant V$).

Given $E\, W_{\tau^*} = E\, G_{\tau^*}$ & $W_{\tau^*} \geqslant G_{\tau^*} \Rightarrow W_{\tau^*} = G_{\tau^*}$

Since $W \geqslant V \geqslant G \Rightarrow W_{\tau^*} = V_{\tau^*} = G_{\tau^*}$.

Claim : $W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$.

(Intuition : Know $\underbrace{W_{\tau^* \wedge n}}_{\text{Mg}} \geq \underbrace{V_{\tau^* \wedge n}}_{\text{super mg}}$.

Guess $\Rightarrow$ Equality .)

Pf : Backward Induction : (1) $W_{\tau^* \wedge N} = V_{\tau^* \wedge N}$ $(\because \tau^* \leq N)$ ✓

② Assume for some n, $\underbrace{W_{\tau^* \wedge (n+1)}}_{\text{mg}} = \underbrace{V_{\tau^* \wedge (n+1)}}_{\text{super mg}}$. $(\& \ W \geq V)$,

$$\Rightarrow E_n \left(W_{\tau^* \wedge (n+1)} \right) = E_n V_{\tau^* \wedge (n+1)}$$

$$\overset{(\text{OST})}{\Rightarrow} \quad W_{\tau^* \wedge n} = E_n V_{\tau^* \wedge (n+1)} \leq V_{\tau^* \wedge n}.$$

Since $W_{\tau^* \wedge n} \geq V_{\tau^* \wedge n} \Rightarrow W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$. QED

Theorem 6.80. *For any $k \in \{0, \dots, N\}$, let $\sigma_k^* = \min\{n \geqslant k \mid V_n = G_n\}$. Then $\boldsymbol{E}_k G_{\sigma_k^*} = \max_{\sigma_k} \boldsymbol{E}_k G_{\sigma_k}$, where the maximum is taken over all finite stopping times σ_k for which $\sigma_k \geqslant k$ almost surely.*

i.e.

$E_k G_{\tau_k^*} \overset{a.s}{\geq} E_k G_{\tau_k}$ for all finite stopping times τ_k such that $\tau_k \geq k$ a.s.

On HW

Theorem 6.81. *Let $V = M - A$ be the Doob decomposition for V, and define $\tau^* = \max\{n \mid A_n = 0\}$. Then τ^* is a stopping time and is the* largest *solution to the optimal stopping problem for G.*

Pf: ① τ^* is a stopping time because A is predictable & inc (Intuition).

Pf: $\{\tau^* \le n\} = \{A_{n+1} > 0\} \in \mathcal{F}_n$ ($\because A$ is predictable).

② NTS τ^* solves the optimal stopping problem

i.e. NTS. $E\, G_{\tau^*} \geq E\, G_\tau$ $\forall$ finite stopping time τ.

Pf: $A_{\tau^*} = 0 \Rightarrow V_{\tau^*} = M_{\tau^*} \boxed{= G_{\tau^*}}$ ← (Why)

I found a better shorter proof of this without relying on what I did for American options. I'll present the proof next class

$$\Rightarrow \; EG_\tau \le EV_\tau \le EM_\tau \qquad (M \ge V \ge G)$$

$$\overset{OST}{=} M_0 \overset{OST}{=} EM_{\tau^*} = EG_{\tau^*}.$$

$\Rightarrow \tau^*$ is a sol to the optimal stopping problem

(3) NTS τ^* is the largest sol to the optimal stopping problem.

Say τ^* is a sol to the optimal stopping problem.

$$\Rightarrow \; EG_{\tau^*} \ge EG_\tau \quad \forall \text{ finite stopping times } \tau.$$

$\Rightarrow$ (Know $G_{\tau^*} = V_{\tau^*}$ & $\boxed{A_{\tau^*} = 0}$ $\leftarrow$ Why.

$A_{\tau^*} = 0 \Rightarrow \tau^* \leq \tau^*$

$\Rightarrow$ τ^* is the largest sol to the optimal stopping problem.

QED

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7. Fundamental theorems of Asset Pricing

Last time : $G = (G_0, \dots, G_N)$.

Snell super mg envelope : $\underline{V_N = G_N}$, $V_n = \max\{G_n, E_n V_{n+1}\}$.

Last time ① V is the <u>smallest</u> super mg $\ni$ $V \geq G$.

Last time ② $\tau^* = \min\{n \mid V_n = G_n\}$. τ^* solves the $\boxed{\text{optimal stopping problem for } G.}$

i.e. $E G_{\tau^*} \geq E G_\tau$ for all finite stopping times τ.

(τ^* is a stopping time).

Theorem 6.81. *Let $V = M - A$ be the Doob decomposition for V, and define $\tau^* = \max\{n \mid A_n = 0\}$. Then τ^* is a stopping time and is the largest solution to the optimal stopping problem for G.*

$M \longrightarrow M_g$

$A \longrightarrow$ pred inc & $A_0 = 0$

Note $\{\tau^* \leq n\} = \{A_{n+1} > 0\} \in \mathcal{F}_n$

($\because A$ is inc) ($\because A$ is pred).

i.e: NTS ① $E G_{\tau^*} \geq E G_\sigma$ $\forall$ finite stopping times σ.

& ② If σ^* is a soln to the opt stopping problem for G then $\sigma^* \leq \tau^*$.

Pf: ① Claim $\boxed{V_{\tau^*} = M_{\tau^*} = G_{\tau^*}}$

ⓐ If $\tau^* = N \longrightarrow V_N = G_N = M_N$. (nothing to check).

(b) Say $n < N$.

$$\mathbb{1}_{\{\tau^* = n\}} E_n V_{n+1} = E_n \left(M_{n+1} - A_{n+1} \right) \mathbb{1}_{\{\tau^* = n\}}$$

$$= \mathbb{1}_{\{\tau^* = n\}} M_n - \mathbb{1}_{\{\tau^* = n\}} A_{n+1}$$

(When $n = \tau^*$
$A_n = 0 \Rightarrow M_n = V_n$
& $A_{n+1} > 0$)

$$< \mathbb{1}_{\{\tau^* = n\}} V_n .$$

But $V_n = \max \left\{ E_n V_{n+1}, G_n \right\}$

We've shown when $n = \tau^*$, $V_n > E_n V_{n+1}$

$\Rightarrow V_n = G_n$. ($A_n = 0 \Rightarrow M_n = V_n = G_n$ when $n = \tau^*$). QED.

Hence we know $V_{\tau^*} = M_{\tau^*} = G_{\tau^*}$.

(2) Check τ^* is optimal: $\forall\, \tau$, $E G_\tau \leq E M_\tau$ ($\because M \geq V \geq G$)

$\overset{OST}{=} E M_0 \overset{OST}{=} E M_{\tau^*} = \underline{E G_{\tau^*}}$

$\Rightarrow \tau^*$ is optimal.

③ NTS $\underline{\underline{\tau^*}}$ is the $\underline{\text{largest}}$ sol to the optimal stopping problem.

Say $\sigma^* \gtrsim \tau^*$ is a solution to the optimal stopping problem for G.

& Say $P(\sigma^* > \tau^*) > 0$.

Then $$E G_{\sigma^*} \leq E V_{\sigma^*} = E(M_{\sigma^*} - A_{\sigma^*})$$

$$= EM_{\sigma^*} - EA_{\sigma^*}$$

$$\stackrel{OST}{=} EM_{\tau^*} - \underline{\underline{EA_{\sigma^*}}}$$

$$= E G_{\tau^*} - E A_{\sigma^*}$$

$$< E G_{\tau^*}. \quad \Rightarrow \sigma^* \text{ can not be optimal!}$$

(by $\;$)

Note $\sigma^* \geq \tau^* \Rightarrow A_{\sigma^*} \geq A_{\tau^*} = 0$

Also, $P(\sigma^* > \tau^*) > 0$

$\Rightarrow P(A_{\sigma^*} > 0) > 0$

$\Rightarrow E A_{\sigma^*} > 0$

QED.

7. Fundamental theorems of Asset Pricing

7.1. Markets with multiple risky assets.

(1) $\Omega = \{1, \ldots, M\}^N$ is a probability space representing N rolls of M-sided dies, and p is a probability mass function on Ω.
(2) The die rolls need not be i.i.d.
(3) Consider a financial market with $d+1$ assets $S^0, S^1, \ldots, S^d$. (S^k_n denotes the price of the k-th asset at time n.)
(4) For $i \in \{1, \ldots, d\}$, S^i is an adapted process (i.e. S^i_n is $\mathcal{F}_n$-measurable).
(5) The 0-th asset S^0 is assumed to be a *risk free* bank/money market:
  (a) Let r_n be an adapted process specifying the interest rate at time n.
  (b) Let $S^0_0 = 1$, and $S^0_{n+1} = (1 + r_n) S^0_n$. (Note S^0 is predictable.)
  (c) Let $D_n = (S^0_n)^{-1}$ be the discount factor (D_n dollars at time 0 becomes 1 dollar at time n).
(6) Let $\Delta_n = (\Delta^0_n, \ldots, \Delta^d_n)$ be the position at time n of an investor in each of the assets $(S^0_n, \ldots, S^d_n)$.
(7) The wealth of an investor holding these assets is given by $X_n = \boxed{\Delta_n \cdot S_n} \stackrel{\text{def}}{=} \sum_{i=1}^{d} \Delta^i_n S^i_n$.

$i = 0$

Q: Say an investor has position $\underline{\Delta}_n$ on the d stocks & Money Market.

What does it mean for the investors wealth to be self financing?

(i.e. → no cash flow injection / removal)

Binomial model

$$X_{n+1} = \Gamma_n S_{n+1} + (X_n - \Gamma_n S_n)(1+r)$$

Wealth at time n: $\displaystyle\sum_{i=0}^{d} \Delta_n^i S_n^i$

Say position at time n is $\underline{\Delta}_n$.

At time $(n+1)$ → my portfolio is now worth $\displaystyle\sum_{i=0}^{d} \Delta_n^i S_{n+1}^i = \Delta_n \cdot S_{n+1}$

Rebalance : (buys / sell some assets)

New position on the d assets is $\underline{\Delta_{n+1}}$

Wealth at time $\underline{n+1}$ is $\Delta_{n+1} \cdot S_{n+1}$

Retained

No ~~go~~ cash injection / withdrawal $\Rightarrow$ $\boxed{\Delta_{n+1} \cdot S_{n+1} = \Delta_n \cdot S_{n+1}}$

Self financing condition.

Definition 7.1. Consider a portfolio whose positions in the assets at time n is Δ_n. We say this portfolio is *self-financing* if Δ_n is adapted, and $\underbrace{\Delta_n \cdot S_{n+1}} = \Delta_{n+1} \cdot S_{n+1}$.

Note: Δ_n is $\mathcal{F}_n$ meas.

Δ_{n+1} is $\mathcal{F}_{n+1}$ meas.

7.2. **First fundamental theorem of asset pricing.**

Definition 7.2. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $\boxed{X_0 = 0}$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Definition 7.3. We say $\tilde{\boldsymbol{P}}$ is a *risk neutral measure* if $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ and $\tilde{\boldsymbol{E}}_n(D_{n+1}S^i_{n+1}) = D_n S^i_n$ for every $i \in \{0, \dots, d\}$.

Theorem 7.4. *The market is arbitrage free if and only if there exists a risk neutral measure.*

Proof that existence of a risk neutral measure implies no-arbitrage.

Recall: $S^0_{n+1} = (1 + r_n) S^0_n$ (bank)

$$\boxed{D_n \stackrel{\text{def}}{=} \frac{1}{S^0_n}}$$

Note $\tilde{E}_n(D_{n+1} S^0_{n+1}) = 1 = D_n S^0_n$

Recall: $\tilde{P}$ is equivalent to P if $P(A) = 0 \iff \tilde{P}(A) = 0$

Pf of FTAP 1 $\Leftarrow$: Assume $\exists$ a RNM.

NTS $\longrightarrow$ No arb.

Pf: X = wealth process of a self fin portfolio.

$$X_n = \Delta_n \cdot S_n \quad \& \quad \boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$$

$\boxed{\text{Q: Is } D_n X_n \text{ a } \tilde{P} \text{ mg?}}$

Yes: Check: $\tilde{E}_n (D_{n+1} X_{n+1}) = \tilde{E}_n (D_{n+1} \Delta_{n+1} \cdot S_{n+1})$

(D_n is predictable)

$$= D_{n+1}\,\tilde{E}_n\left(\Delta_{n+1} \cdot S_{n+1}\right)$$

$$\overset{\text{self financing}}{=} D_{n+1}\,\tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right)$$

$$= D_{n+1}\,\tilde{E}_n\left(\sum_{i=0}^{d} \Delta_n^i\, S_{n+1}^i\right)$$

$$= \sum_{i=0}^{d} \Delta_n^i\, \tilde{E}_n\left(\underbrace{D_{n+1} S_{n+1}^i}\right) \overset{\text{RNM}}{=} \sum_{i=0}^{d} \Delta_n^i\, D_n S_n^i$$

$$= D_n\left(\Delta_n \cdot S_n\right) = D_n X_n$$

QED (Claim)

Have shown: X_n = wealth process of a self fin Port

$$\Rightarrow D_n X_n \text{ is a } \tilde{P} \text{ mg.}$$

Pf there is No Arb:

Say $X_0 = 0$, X is wealth proc of a self fin port.

$$X_N \geq 0.$$

$$\Rightarrow D_n X_n \text{ is a } \tilde{P} \text{ mg.} \Rightarrow \tilde{E}(D_N X_N) \overset{mg}{=} \tilde{E}(D_0 X_0) = 0$$

$$\Rightarrow D_N X_N = 0 \text{ a.s. } (\because D_N X_N \geq 0)$$

$$\Rightarrow \quad X_N = 0 \quad \text{a.s} \qquad (\because D_N > 0).$$

QED.

Last time : d stocks. Price : $S^1 \cdots S^d$ (adapted processes)

M. M. S^0 (predictable)

$$S^0_{n+1} = (1+r_n)\, S^0_n \qquad (r_n \to \text{interest rate, adapted})$$

Discount factor : $D_n = \dfrac{1}{S^0_n}$ $\quad (\Leftrightarrow D_n S^0_n = 1)$

Self financing : $\Delta_n = (\Delta^0_n, \cdots \Delta^d_n)$ (position in the $d+1$ assets)

$$\Delta_n \cdot S_n = \text{wealth at time } n = \sum_0^d \Delta^i_n S^i_n .$$

Self fin : $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

7.2. **First fundamental theorem of asset pricing.**

Definition 7.2. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Definition 7.3. We say $\tilde{\boldsymbol{P}}$ is a *risk neutral measure* if $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ and $\tilde{\boldsymbol{E}}_n(D_{n+1}S^i_{n+1}) = D_n S^i_n$ for every $i \in \{0, \dots, d\}$.

Theorem 7.4. *The market defined in Section 7.1 is arbitrage free if and only if there exists a risk neutral measure.*

Lemma 7.5. *If $\tilde{\boldsymbol{P}}$ is a risk neutral measure, then the discounted wealth of any self financing portfolio is a $\tilde{\boldsymbol{P}}$-martingale.*

Proof that existence of a risk neutral measure implies no-arbitrage. (last time).

$\longrightarrow$ did last time. $\Longrightarrow$

Goal now: No arb $\Longrightarrow$ $\exists$ a RNM.

Lemma 7.6. *Suppose the market has no arbitrage, and X is the wealth process of a self-financing portfolio. If for any n, $X_n = 0$ and $X_{n+1} \geqslant 0$, then we must have $X_{n+1} = 0$ almost surely.*

Pf: Say we had $X_n = 0$, $X_{n+1} \geqslant 0$ & $P(X_{n+1} > 0) > 0$.

then move all \$ to bank

Get $\geqslant 0$ wealth at time N

& $P(0 <$ wealth at time $N) > 0$ } Not allowed by the no arb assumption

QED

Lemma 7.7. *Suppose we find an equivalent measure $\tilde{\boldsymbol{P}}$ such that whenever $\Delta_n \cdot S_n = 0$, we have $\tilde{\boldsymbol{E}}_n(\Delta_n \cdot S_{n+1}) = 0$, then $\tilde{\boldsymbol{P}}$ is a risk neutral measure.*

Pf: NTS $\tilde{E}_n\left(D_{n+1} S^i_{n+1}\right) = D_n S^i_n \quad \forall\, i \in \{\cancel{0}, \dots d\}$

First show this for $i = 1$ (For other i the proof is identical)

At time n → buy $\underline{1 \text{ share}}$ of 1^{st} stock. (Costs S^1_n)

→ sell S^1_n cash. $\left(= \dfrac{S^1_n}{S^0_n}\right.$ shares of M.M at time $n\left.\right)$.

i.e. Chose $\Delta_n = \left(-\dfrac{S^1_n}{S^0_n}, 1, 0 \dots 0\right)$

Note $\Delta_n \cdot S_n = \left(-\frac{S_n^1}{S_n^0}, 1, 0 \cdots \right) \cdot \left(S_n^0, S_n^1, \cdots \right)$

$$= -\frac{S_n^1}{\cancel{S_n^0}} \cancel{S_n^0} + S_n^1 = 0$$

$\Rightarrow \Delta_n \cdot S_n = 0$. Hence, by assumption, $\tilde{E}_n \left(\Delta_n \cdot S_{n+1} \right) = 0$

$$\Rightarrow \tilde{E}_n \left(-\frac{S_n^1}{S_n^0} \cdot S_{n+1}^0 + 1 \cdot S_{n+1}^1 + 0 \cdots \right) = 0$$

$$\Rightarrow -\frac{S_n^1}{S_n^0} \cdot S_{n+1}^0 + \tilde{E}_n S_{n+1}^1 = 0$$

$$\Rightarrow \tilde{E}_n S^1_{n+1} = \frac{S^1_n}{S^0_n} \cdot S^0_{n+1}$$

$$\boxed{\Rightarrow \tilde{E}_n \left(D_{n+1} S^1_{n+1} \right) = D_n S^1_n} \Rightarrow D_n S^1_n \text{ is a } \tilde{P} \text{ mg}$$

Repeat for all i $\Rightarrow D_n S^i_n$ is a $\tilde{P}$ mg $\forall i$

$\Rightarrow \tilde{P}$ is a RNM

QED.

Lemma 7.8. *Suppose* $\tilde{p}$ *is a probability mass function such that* $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\tilde{p}_2(\omega_1, \omega_2) \cdots \tilde{p}_N(\omega_1, \dots, \omega_N)$. *If* X_{n+1} *is* $\mathcal{F}_{n+1}$-*measurable, then*

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \sum_{i=1}^{M} \tilde{p}_{n+1}(\omega', j) X_{n+1}(\omega', j)\,, \qquad \textit{where} \qquad \omega' = (\omega_1, \dots, \omega_n)\,, \omega = (\omega', \cancel{\omega_{n+1}}, \omega_{n+1}, \dots, \omega_N)$$

Lemma 7.9. *Define* $\bar{Q} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i \geqslant 0 \ \forall i \in \{1, \ldots, M\}\}$, *and* $\mathring{Q} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i > 0 \ \forall i \in \{1, \ldots, M\}\}$. *Let* $V \subseteq R^M$ *be a subspace.*

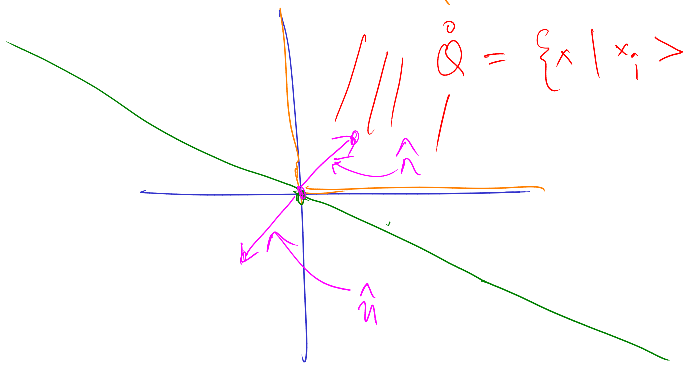
(1) $V \cap \bar{Q} = \{0\}$ *if and only if there exists* $\hat{n} \in \mathring{Q}$ *such that* $|\hat{n}| = 1$ *and* $\hat{n} \perp V$. $\longleftarrow$ $(\hat{n} \cdot v = 0 \ \forall v \in V)$

(2) *The normal vector* $\hat{n}$ *is unique if and only if* $\dim(V) = M - 1$.

Remark 7.10. This is a special case of the *Hyperplane separation theorem* used in convex analysis.

$\bar{Q} = \{x \mid x_i \geqslant 0\}$

$\mathring{Q} = \{x \mid x_i > 0 \ \forall i\}$



Proof of Theorem 7.4. Assume no arb. NTS $\exists$ a RNM.

① Construct the RNM using a PMF $\tilde{P}$ of the form

$$\tilde{P}(\omega) = \tilde{P}_1(\omega_1)\, \tilde{P}_2(\omega_1, \omega_2) \cdots \tilde{P}_N(\omega_1, \cdots \omega_N)$$

& will find each $\tilde{P}_i$.

② Pick $n \in \{0, \cdots N-1\}$. Will find $\tilde{P}_n$

③ Know No arb $\Rightarrow$ No arb at time n.

i.e. If $\Delta_n \cdot S_n = 0$ & $\Delta_n \cdot S_{n+1} \geq 0$

& $\Delta_{n+1} \cdot S_{n+1} = \Delta_n \cdot S_{n+1}$

$$\Rightarrow \underline{\Delta_{n+1} \cdot S_{n+1} = 0}$$

Fix $\omega' = (\omega_1, \cdots \omega_n)$. Write $\Delta_{n+1} = \Delta_{n+1}(\omega', \omega_{n+1})$.

Let $$V = \left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', 2) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \middle| \; \Delta_n(\omega') \cdot S_n(\omega') = 0 \right\}$$

Think of $V \subseteq \mathbb{R}^M$. Note V is a subspace of $\mathbb{R}^M$.

No arb: $V \cap \bar{Q} = \{0\}$.

$$\bar{Q} = \{ v \mid v_i \geq 0 \}$$

By lemma $\exists$ a $\boxed{\text{normal vector } \hat{n}}$ $\in \mathring{Q}$

Will use $\hat{n}$ to construct RNM.

$$\hat{n} = \hat{n}(\omega') = \begin{pmatrix} \hat{n}_1(\omega') \\ \vdots \\ \hat{n}_M(\omega') \end{pmatrix}.$$

Set: $$\tilde{\phi}_n(\omega', j) = \frac{\hat{n}_j(\omega')}{\sum \hat{n}_i(\omega')}$$

Claim (You check): $\underbrace{\tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right)} = 0$

(By lemma $\Rightarrow \tilde{P}$ is a RNM $\Rightarrow$ QED.)

Last time: 1^{st} fund thm: No arb $\iff$ $\exists$ a RNM

Pf from last time: Assume no arb. NTS $\exists$ a RNM.

Going to construct $\tilde{P}$ with PMF $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\,\tilde{p}_2(\omega_1,\omega_2) \cdots \tilde{p}_N(\omega_1 \cdots \omega_N)$.

Find $\tilde{p}_n$. $\longrightarrow$ Fix $\omega' = (\omega_1, \ldots \omega_n)$

$$\text{Let } V = \left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \,\middle|\, \underbrace{\Delta_n(\omega') \cdot S_n(\omega') = 0}_{\text{Net worth 0 at time } n} \right\}$$

j^{th} coordinate = net worth at time $n+1$ if $(n+1)^{th}$ die roll is j.

You check: $V \subseteq \mathbb{R}^M$ is a subspace (HW, please check)

$\bar{Q} = \{v \in \mathbb{R}^M \mid v_i \geq 0\}$, $\mathring{Q} = \{v \in \mathbb{R}^M \mid v_i > 0\}$

Note: No arb $\Rightarrow V \cap \bar{Q} = \{0\}$.

Separation lemma $\Rightarrow \exists\, \hat{n} \in \mathring{Q}$ + $|\hat{n}| = 1$ & $\hat{n} \perp V$
(i.e $\hat{n} \cdot v = 0 \;\forall v \in V$).

Use $\hat{n}$ to define $\tilde{p}_{n+1}(\omega', j)$: $\boxed{\tilde{p}_{n+1}(\omega', j) = \dfrac{\hat{n}_j}{\sum_{i=1}^{M} \hat{n}_i}}$

$$\Rightarrow \tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right)(\omega') = \sum_{j=1}^{M} \Delta_n(\omega') \cdot S_{n+1}(\omega', j) \cdot \tilde{p}_{n+1}(\omega', j)$$

$$= \frac{1}{\sum_{i=1}^{M} \hat{n}_i} \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \cdot \hat{n}$$

$\perp V$.

$\in V$

$= 0$

Lemma from last time $\Rightarrow$ $\tilde{P}$ is a RNM.

QED

7.3. Second fundamental theorem.

Definition 7.11. A market is said to be *complete* if every derivative security can be hedged.

Theorem 7.12. *The market defined in Section 7.1 is complete and arbitrage free if and only if there exists a* unique *risk neutral measure.*

(i.e. For every security $\exists$ a replicating portfolio)

Lemma 7.13. *The market is complete if and only if for every $\mathcal{F}_{n+1}$-measurable random variable X_{n+1}, there exists a (not necessarily unique) $\mathcal{F}_n$ measurable random vector $\Delta_n = (\Delta_n^0, \ldots, \Delta_n^d)$ such that $X_{n+1} = \Delta_n \cdot S_{n+1}$.*

Pf: Say first $\forall X_{n+1}$ $\exists \Delta_n$ s.t. $X_{n+1} = \Delta_n \cdot S_{n+1}$

(take position Δ_n at time n
Wealth at time $n+1$ $= \Delta_n \cdot S_{n+1}$)

Consider any security that pays G_N at time N.

NTS $\exists$ a rep portfolio. $\longrightarrow$ $\exists$ a self fin trading strat with final wealth G_N.

① By assumption $\exists \Delta_{N-1}$ ($\mathcal{F}_{N-1}$ meas) s.t. $\Delta_{N-1} \cdot S_N = G_N$.

② $\Delta_{N-1} \cdot S_{N-1}$ is $\mathcal{F}_{N-1}$ meas $\Rightarrow$ $\exists \Delta_{N-2}$ ($\mathcal{F}_{N-2}$-meas) s.t. $\Delta_{N-2} \cdot S_{N-1} = \Delta_{N-1} \cdot S_{N-1}$

⋮

(n) Backward induction: $\forall n$, $\exists \Delta_n$ ($\mathcal{F}_n$-meas) s.t. $\underbrace{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}_{\text{self financing condition.}}$

& $\Delta_{N-1} \cdot S_N = G_N$.

$\Rightarrow (\Delta_n)$ gives me a self-financing trading strat that replicates the security.

✓

Pf of converse is similar (please check!)

Proof of Theorem 7.12 NTS complete + arb free $\Longleftrightarrow$ unique RNM.

Recall how we constructed $\tilde{P}$ in the first fund thm.

Fix n, $\omega' = (\omega_1, \ldots, \omega_n)$.

$$V = \left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \;\middle|\; \Delta_n(\omega') \cdot S_n(\omega') = 0 \right\}$$

How did we construct RNM: Picked $\hat{n} \perp V$ & $\hat{n} \in \mathring{Q}$ (all +ve coords)

Note any $\hat{n}$ s.t. $\hat{n} \perp V$ & $\hat{n} \in \mathring{Q}$ gives a RNM

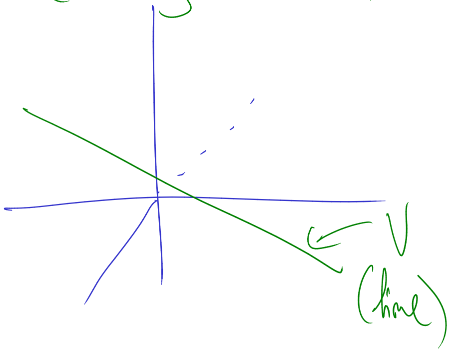
by $$\tilde{f}_n(\omega', j) = \frac{\hat{n}_j}{\sum_{i=1}^{M} \hat{n}_i}$$

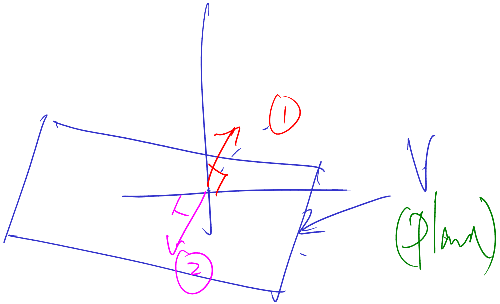
Hence Unique RNM $\iff$ $\exists$ a unique $\hat{n} \in \mathring{Q}$ s.t. $|\hat{n}| = 1$ & $\hat{n} \perp V$

$$\iff \dim(V) = M - 1 \quad \& \quad \underbrace{V \cap \bar{Q} = \{0\}}_{\text{no arbitrage}}$$

$$\iff \mathbb{R}^M = \operatorname{span}\left\{ \begin{pmatrix} \Delta_n(\omega') \cdot S_{n+1}(\omega', 1) \\ \vdots \\ \Delta_n(\omega') \cdot S_{n+1}(\omega', M) \end{pmatrix} \;\middle|\; \cancel{\Delta_n(\omega') \cdot S_n(\omega') = 0} \right\}$$

(∞'ly many normal vectors)





$\iff$ any δ_{n+1} RV can be $\underbrace{\text{written in the form } \Delta_n \circ S_{n+1}}_{\Updownarrow}$ & $\boxed{\text{no amb}}$

completeness (by lemma).

$\Longleftrightarrow$ completeness & no amb

Q.E.D.

7.4. **Examples and Consequences.**

Proposition 7.14. *Suppose the market model Section 7.1 is complete and arbitrage free, and let* $\tilde{\boldsymbol{P}}$ *be the unique risk neutral measure. If* $D_n X_n$ *is a* $\tilde{\boldsymbol{P}}$ *martingale, then* X_n *must be the wealth of a self financing portfolio.*

Remark 7.15. We've already seen in Lemma 7.5 that if a (not necessarily unique) risk neutral measure exists, then the discounted wealth of any self financing portfolio must be a martingale under it.

Remark 7.16. All pricing results/formulae we derived for the Binomial model that only relied on the analog of Proposition 7.14 will hold in complete arbitrage free markets.

Pf: Assume $D_n X_n$ is a $\tilde{P}$ mg (i.e. $\tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n$).

Reminder: ① $D_n = \dfrac{1}{S^0_n}$ ($S^0_n \to$ Bank process)

② Say $\Delta_n \to$ trading strat. (adapted)

$\Delta_n = (\Delta^0_n, \Delta^1_n, \dots \Delta^d_n)$

Wealth at time $n = \Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$

$$\boxed{\text{Self financing: } \Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$$

Assume $D_n X_n$ is a $\tilde{P}$ mg.

Need to find Δ_n $\ni$ $X_n = \Delta_n \cdot S_n$

& $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

Note: Since the market is complete, for any $\mathcal{F}_{n+1}$ meas. R.V. $\nearrow X_{n+1}$

$\exists$ an $\mathcal{F}_n$ meas R.V Δ_n $\ni$ $\boxed{X_{n+1} = \Delta_n \cdot S_{n+1}}$

So by completeness know $\exists$ Δ_n adapted $\ni$ $X_{n+1} = \Delta_n \cdot S_{n+1}$

Need: (1) $X_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

& (2) $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

(Note (2) $\Rightarrow$ (1) since $X_{n+1} = \Delta_n \cdot S_{n+1}$)

Prove (2): know $\tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n$ (linear)

Hence $D_n X_n = \tilde{E}_n\left(D_{n+1} X_{n+1}\right) - D_{n+1}\tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right)$

$\underline{\underline{D_n}}\;\Delta_{n-1}\cdot S_{n}$

$$= \sum_{i=0}^{d} \Delta_n^i \,\tilde{E}_n\left(P_{n+1} S_{n+1}^i\right)$$

(Def of RNM) $$= \sum_{i=0}^{d} \Delta_n^i \left(D_n S_n^i\right) = \underline{\underline{D_n}}\;\Delta_n \cdot S_n$$

$$\Rightarrow \Delta_{n-1}\cdot S_{n} = \Delta_n \cdot S_n. \Rightarrow 2$$

QED

Question 7.17. *Consider a market consisting of a bank with interest rate r, and two stocks with price processes S^1, S^2. At each time step we flip two independent coins. The price of the i-th stock ($i \in \{1, 2\}$) changes by factor u_i, or d_i depending on whether the i-th coin is heads or tails. When is this market arbitrage free? When is this market complete?*

2 stocks

(2 indep coins flipped at each time)

Guess: Arb free & complete $\iff$
$$\left.\begin{aligned} &0 < d_1 < 1+r < u_1 \\ \& \ &0 < d_2 < 1+r < u_2. \end{aligned}\right\}$$

doesn't work

Claim will $\Rightarrow$ arb free (not completeness)

Claim: this Market can $\boxed{\text{NEVER}}$ be made complete & arb free

Consider what happens after the first time period:

Look for the RNM. 2 coins $\longrightarrow$ outcomes HH, HT, TH, TT.

RNM probabilities: $\tilde{p}(HH)$, $\tilde{p}(HT)$, $\tilde{p}(TH)$, $\tilde{p}(TT)$.

Want $\frac{1}{(1+r)}\tilde{E}\, S_1^1 = S_0^1 \iff \tilde{E}\, S_1^1 = (1+r)\, S_0^1$

& $\frac{1}{1+r}\tilde{E}\, S_1^2 = S_0^2 \iff \tilde{E}\, S_1^2 = (1+r)\, S_0^2$

$$\Leftrightarrow (1+r)\cancel{S_0^1} = u_1 \cancel{S_0^1}\left(\tilde{p}(HH) + \tilde{p}(HT)\right) + d_1 \cancel{S_0^1}\left(\tilde{p}(TH) + \tilde{p}(TT)\right) \quad (1)$$

$$\& \quad (1+r)\cancel{S_0^2} = u_2 \cancel{S_0^2}\left(\tilde{p}(HH) + \tilde{p}(TH)\right) + d_2 \cancel{S_0^2}\left(\tilde{p}(HT) + \tilde{p}(TT)\right) \quad (2)$$

$$\tilde{p}(HH) + \tilde{p}(HT) + \tilde{p}(TH) + \tilde{p}(TT) = 1 \quad (3)$$

3 equations.

Unknowns: $\tilde{p}(HH), \tilde{p}(HT), \tilde{p}(TH), \tilde{p}(TT)$ (4 unknowns!)

Solve with the constraints

$$0 \le \tilde{p}(HH) \le 1$$

$$0 \le \tilde{p}(HT) \le 1$$

$$0 \le \tilde{p}(TH) < 1 \quad \& \quad 0 < \tilde{p}(TT) < 1$$

Ill constrained system $\rightarrow$ if solutions exist they will NEVR be unique!

(Market can never be complete & arb free)

Q: Say $0 < d_1 < 1+r < u_1$
& $0 < d_2 < 1+r < u_2$ } Does this $\Rightarrow$ existence of a RNM.

Yes: Let $\tilde{p}_1 = \dfrac{1+r-d_1}{u_1-d_1}$, $\tilde{q}_1 = \dfrac{u_1-(1+r)}{u_1-d_1}$

$\tilde{p}_2 = \dfrac{1+r-d_2}{u_2-d_2}$, $\tilde{q}_2 = \dfrac{u_2-(1+r)}{u_2-d_2}$

& Choose $\tilde{p}(HH) = \tilde{p}_1 \cdot \tilde{p}_2$

$\tilde{p}(HT) = \tilde{p}_1 \tilde{q}_2$

$\tilde{p}(TH) = \tilde{q}_1 \tilde{p}_2$

$\tilde{p}(TT) = \tilde{q}_1 \tilde{q}_2$ & check this gives a RNM.

① $\underline{V} \cap \underline{\bar{Q}} = \{0\} \iff \exists\, \hat{n} \in \underline{\underline{\mathring{Q}}}$ s.t. $\underline{\underline{\hat{n}}} \perp V$.

(Stated but did not prove) (Hyp sep thm)

② Say $\exists$ a $\underline{\text{RNM}}$ NTS no arb

Say $X_0 = 0$, $\underline{\underline{X_N \geq 0}}$. Know $(D_n X_n)$ is a mg under $\underline{\underline{\tilde{P}}}$

$\Rightarrow E D_N X_N = E D_0 X_0 = 0$

$\Rightarrow X_N D_N \geq 0$ & $E(D_N X_N) = 0 \Rightarrow D_N X_N = 0$

Last time:

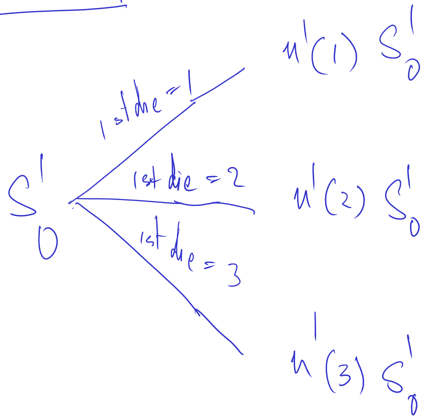
2 indep coins tossed at each time step.

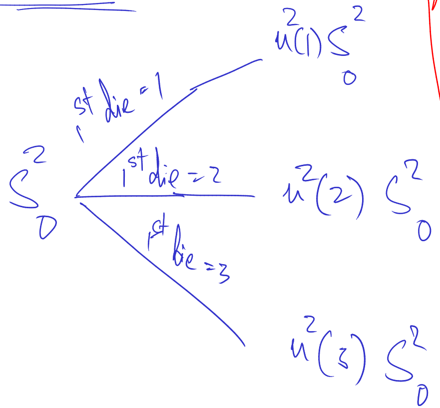
→ conditions for this market to be arb free.

→ <u>Never</u> complete!

Question 7.18. *Consider now repeated rolls of a 3-sided die and for $i \in \{1, 2\}$, let $Z^i_n = u^i(j)$ if the n-th die rolls j. Suppose $S^i_{n+1} = S^i_n Z^i_{n+1}$. Find conditions when this market is complete and arbitrage free.*

And a bank with interest rate $r > -1$





First analyze case $N = 1$ (1 period case)

FTAP 2 $\exists!$ Unique RNM $\iff$ complete & arb free.

Solve for the RNM (time 1)

Want $\tilde{p}(1)$, $\tilde{p}(2)$, $\tilde{p}(3)$ $\ni$ ① $\tilde{E}\left(\frac{S_1^1}{1+r}\right) = S_0^1$

& ② $\tilde{E}\left(\frac{S_1^2}{1+r}\right) = S_0^2$

$\boxed{\tilde{p}(i) = \text{RN prob that the first die rolls } i.}$

$\Rightarrow$ ① $\Rightarrow \sum_{i=1}^{3} \tilde{p}(i) S_1^1(i) = (1+r) S_0^1$

② $\Rightarrow \sum_{i=1}^{3} \tilde{p}(i) S_1^2(i) = (1+r) S_0^2$

& ③ Want $\sum_{i=1}^{3} \tilde{p}(i) = 1$

$\boxed{(\text{\& ④ } \tilde{p}(i) > 0 \ \forall i)}$

$$(1) \,\&\, (2) \iff (1+r)\cancel{S_0^j} = \sum_{i=1}^{3} \tilde{p}(i)\, S_1^j(i) = \sum_{i=1}^{3} \tilde{p}(i)\, u^j(i)\cancel{S_0^j} \quad (\text{for } j = 1 \,\&\, 2)$$

$\Rightarrow$ Equations are

$$\tilde{p}(1) + \tilde{p}(2) + \tilde{p}(3) = 1 \quad \cdots \quad (1)$$

$$\tilde{p}(1)\, u^1(1) + \tilde{p}(2)\, u^1(2) + \tilde{p}(3)\, u^1(3) = 1+r \quad \cdots \quad (2)$$

$$\tilde{p}(1)\, u^2(1) + \tilde{p}(2)\, u^2(2) + \tilde{p}(3)\, u^2(3) = 1+r \quad \cdots \quad (3)$$

$$\iff \begin{pmatrix} 1 & 1 & 1 \\ \longleftarrow & u^1 & \longrightarrow \\ \longleftarrow & u^2 & \longrightarrow \end{pmatrix} \begin{pmatrix} \tilde{p}(1) \\ \tilde{p}(2) \\ \tilde{p}(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 1+r \\ 1+r \end{pmatrix}$$

Let $A = \begin{pmatrix} \leftarrow & u^1 & \rightarrow \\ \leftarrow & u^2 & \rightarrow \end{pmatrix}$

Q: When does $A \begin{pmatrix} \uparrow \\ \tilde{p} \\ \downarrow \end{pmatrix} = \begin{pmatrix} 1+r \\ 1+r \end{pmatrix}$ have a unique sol?

$\det(A) \neq 0 \Rightarrow$ unique sol!

Also need to ~~m~~ ensure $\tilde{p}(i) > 0 \quad \forall i$

Given u^j, Can explicitly compute $A^{-1} \begin{pmatrix} 1+r \\ 1+r \end{pmatrix}$ & check $\tilde{p}(i) > 0 \ \forall i$ (one period)

$\Rightarrow$ Complete & arb free

(2) More than one period:

Note $S^i_{n+1} = S^i_n \underbrace{Z^i_{n+1}}$ only depends on the $(n+1)^{th}$ die roll.

Choose the RNM with iid die rolls.

Want $\tilde{E}_n\left(\dfrac{S^i_{n+1}}{1+r}\right) = S^i_n$

Note: $\tilde{E}_n(S^i_n Z^i_{n+1}) = S^i_n \tilde{E}_n(Z^i_{n+1})$

$\Rightarrow$ Need $\tilde{E}_n\left(Z^i_{n+1}\right) = 1+r \quad \forall i$

iid die rolls (RNM) $\Leftrightarrow$ $\underline{\tilde{E}\, Z^i_{n+1}} = 1+r$

Let $\tilde{p}_{n+1}(j)$ = RN prob that the $(n+1)^{th}$ die rolls j.

$$\Rightarrow \tilde{E}\, Z^i_{n+1} = \sum_{j=1}^{3} \tilde{p}_{n+1}(j)\, u^i(j)$$

Same eqn's as before & solve as before!

8. Black-Scholes Formula

$C(t) = C(0)e^{rt}$

(1) Suppose now we can trade *continuously in time.*
(2) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.
(3) The $\boxed{\textit{continuously compounded interest rate}}$ is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$).
(4) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.
(5) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).
(6) The price at time t of a European call with maturity T and strike K is given by

x = spot price of stock

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

$(\tau = T - t)$

$$\text{where} \quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,.$$

N = CDF of the Normal dist.

(7) We will derive this as the limit of the $\boxed{\text{Binomial model as } N \to \infty}$.

(4) ① Law of Large #'s

$X_1, X_2, X_3, \ldots$ (iid).

$\underline{E} X_i = \mu$

Say we play N times

$\left\{ \begin{array}{l} \text{Avg winnings: } \underbrace{\frac{1}{N} \sum_{k=1}^{N} X_k}_{\downarrow N \to \infty} \\ \text{Expected winngs} = \mu \end{array} \right.$

8.1. **Law of large numbers.** Now consider infinitely many i.i.d. random variables $X_1, X_2, \dots$.

Theorem 8.1 (Weak law of large numbers). *Suppose* $\boldsymbol{E}X_n = \mu$ *and* $\operatorname{Var} X_n = \sigma^2 < \infty$, *and let* $S_n = \underbrace{\sum_1^n X_k}_{\text{total winnings after } n \text{ plays.}}$. *Then* $\operatorname{Var}(S_n/n) \to 0$, *and hence for any* $\varepsilon > 0$, $\lim_{n\to\infty} \boldsymbol{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = 0$.

Lemma 8.2 (Chebychev's inequality). *For any* $\varepsilon > 0$, $\boldsymbol{P}(X > \varepsilon) \leqslant \frac{1}{\varepsilon}\boldsymbol{E}|X|$.

Pf: Note $\varepsilon \mathbb{1}_{\{X > \varepsilon\}} \leq |X|$

$\Rightarrow \varepsilon E \mathbb{1}_{\{X > \varepsilon\}} \leq E|X|$

$\Rightarrow P(X > \varepsilon) \leq \frac{1}{\varepsilon} E|X|$. QED.

Proof of Theorem 8.1 $EX_n = \mu$, $E(X_n - \mu)^2 = \sigma^2$. X_n's are i.i.d.

$S_n = \sum_1^n X_k$

① NTS $\mathrm{Var}\left(\frac{S_n}{n}\right) \longrightarrow 0$.

Pf: Note $\mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(S_n) = \frac{1}{n^2}\sum_1^n \mathrm{Var}(X_k) = \frac{\sigma^2}{n}$.

$\Rightarrow \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0$

($\because$ X & Y are indep $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$)

② NTS $\forall \varepsilon > 0$, $P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \xrightarrow{n\to\infty} 0$

Pf: $P\left(\left|\frac{S_n}{n}-\mu\right|>\varepsilon\right)=P\left(\left(\frac{S_n}{n}-\mu\right)^2>\varepsilon^2\right)$

$$\overset{\text{Chebyshev}}{\leq} \frac{1}{\varepsilon^2} \underbrace{E\left(\frac{S_n}{n}-\mu\right)^2}_{=\operatorname{Var}\left(\frac{S_n}{n}\right)}$$

$$=\frac{1}{\varepsilon^2}\cdot\frac{\sigma^2}{n} \longrightarrow 0$$

QED.

Theorem 8.3 (Strong law of large numbers)**.** *Under the same assumptions as Theorem 8.1,* $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ *almost surely.*

Expected return

avg winnings after n plays

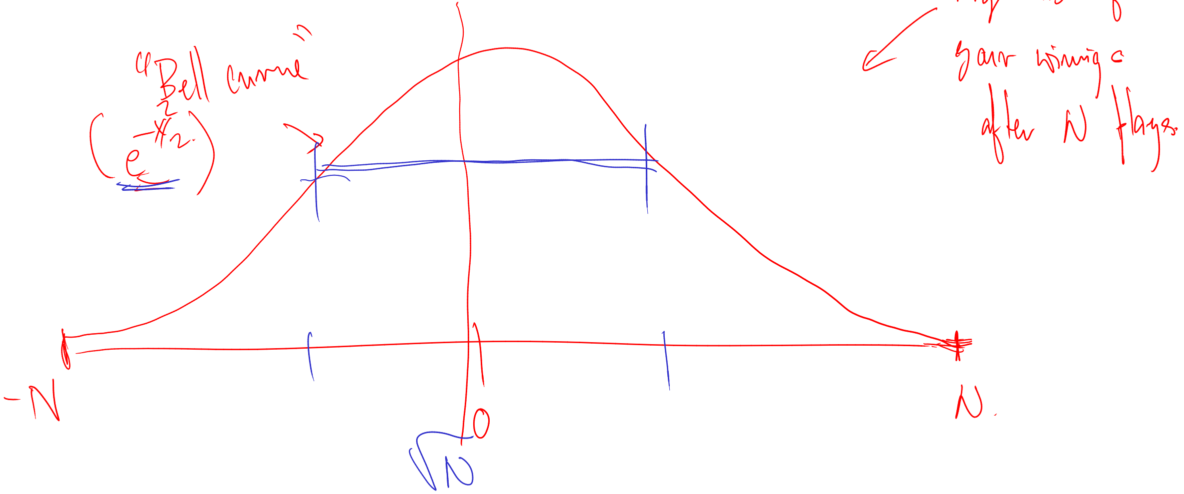
(Pf is harder)

Play a fair slot machine N times.

$X_k \to$ outcome of kth slot. $\quad X_k = 1$ with prob $1/2$

(iid) $\quad X_k = -1$ with prob $1/2$

$E X_k = 0$



8.2. Central limit theorem.

Theorem 8.4. *Let X_n be a sequence of $\mathbb{R}^d$ valued, i.i.d. random variables be such that $\boldsymbol{E}X_n^i = \cancel{\mu_i}\ 0$ and $\operatorname{cov}(X_n^i, X_n^j) = \Sigma_{i,j}$. Let $S_N = \sum_1^N X_n$. Then $(S_N \cancel{-N\mu})/\sqrt{N}$ converges weakly to $\mathcal{N}(\cancel{\mu}\ 0, \Sigma)$.*

Definition 8.5. We say a sequence of random variables Y_n converges weakly to a random variable Z if $\boldsymbol{E}f(Y_n) \to \boldsymbol{E}f(Z)$ for every bounded continuous function f.

Definition 8.6. Let $\mu \in \mathbb{R}^d$, and Σ be a $d \times d$ *covariance matrix* (positive semi-definite, symmetric). $((\Sigma x) \cdot x \geqslant 0 \ \forall x \in \mathbb{R}^d)$

(1) $\mathcal{N}(\mu, \Sigma)$ denotes a *normally distributed random variable* with mean μ and covariance matrix Σ.

(2) When Σ is invertible, the probability density function of $\mathcal{N}(\mu, \Sigma)$ is $\dfrac{1}{(2\pi \det(\Sigma))^{d/2}} \exp\Bigl(-\dfrac{1}{2}(x-\mu) \cdot \Sigma^{-1}(x-\mu)\Bigr)$

(3) When $d = 1$, $\Sigma = \sigma^2$ (Variance) the PDF of $\mathcal{N}(\mu, \sigma^2)$ is $\dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$.

(4) When $\mu = 0$, $\sigma = 1$, $\mathcal{N}(0,1)$ is called the *standard normal*, and its PDF is the *Gaussian* $G(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Definition 8.7. We say p is the probability density function (PDF) of a d-dimensional random variable X if $\boldsymbol{P}(X \in A) = \int_A p(x)\, dx$ for all cubes $A \subseteq \mathbb{R}^d$.

Remark 8.8. Equivalently, p is the PDF of X if $\boldsymbol{E}f(X) = \int_{\mathbb{R}^d} f(x)p(x)\, dx$ for every bounded continuous function f.

Remark 8.9. We will prove Theorem 8.4 during the course of the construction of Brownian motion.

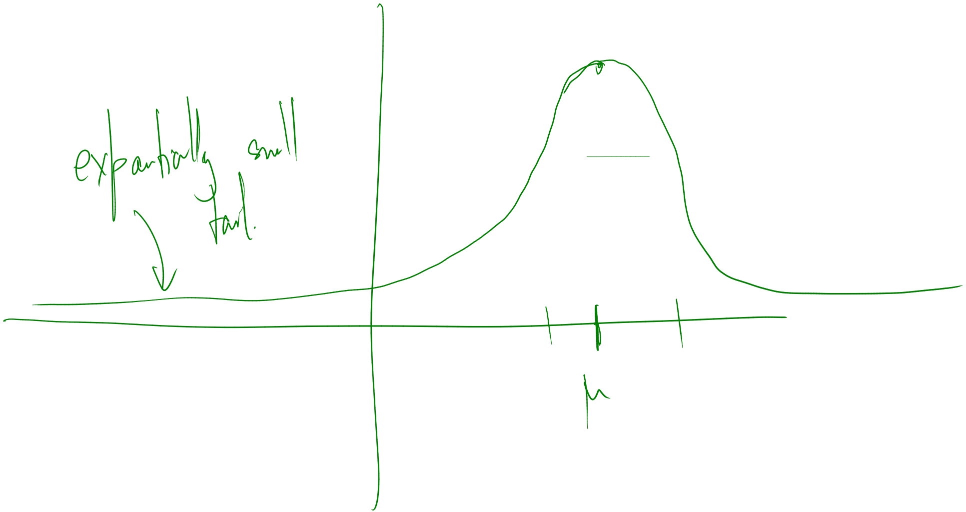
S.LLL: $EX_n = 0$. $\frac{S_n}{n} \longrightarrow \mu = 0$ a.s.

CLT: $EX_n = 0$: $\frac{S_n}{\sqrt{n}} \xrightarrow{\text{"weakly"}} \underbrace{N(0, \Sigma)}_{\text{Normally dist RV mean } 0\text{, cov. } \Sigma}$

$Y_n \to Z$ weakly if $Ef(Y_n) \longrightarrow Ef(Z)$ $\forall$ bdd cts fns f.

$\Leftrightarrow$ for cts RVs, $\lim_{n\to\infty} P(Y_n \in A) = P(Z \in A)$ ($\forall A \subseteq \mathbb{R}$ intervals)

$$G_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \text{(Gaussian)}$$



X is a $\mathbb{R}$ valued RV.

p is the PDF of X if $\forall\, a, b \in \mathbb{R}$, $P(X \in (a,b)) = \int_a^b p(x)\, dx$

$\Leftrightarrow$ $\forall$ bdd cts fn f, $E f(X) = \int_{\mathbb{R}} f(x)\, p(x)\, dx$

(Note: For disc RV's, p is the PMF.

Y is a RV. $E f(Y) = \sum f(y_i)\, P(Y = y_i)$

(Range $Y = \{y_1, \dots, y_n\}$))

8.3. **Brownian motion.**

- Suppose now $X_1, X_2, \dots$ are i.i.d. $\mathbb{R}$ valued random variables.
- Use $\tilde{\boldsymbol{P}}$ to denote the probability measure, and $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ to denote the associated expectation / conditional expectation.
- Assume $\tilde{\boldsymbol{E}} X_n = 0$, and $\tilde{\boldsymbol{E}} X_n^2 = 1$.

Theorem 8.10. *Let $W_n^N = \frac{1}{\sqrt{N}} S_n = \frac{1}{\sqrt{N}} \sum_1^n X_k$. Then $\lim_{N \to \infty} W_{\lfloor Nt \rfloor}^N$ exists almost surely.*

Theorem 8.11. *(1) The function $t \mapsto W_t$ is continuous almost surely, and $W_0 = 0$.*
(2) If $0 = t_0 < t_1 < \cdots t_n$, then $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, $\dots$, $W_{t_n} - W_{t_{n-1}}$ are independent and $W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$.

Remark 8.12. Typically one changes the probability space to ensure the function $t \mapsto W_t$ is continuous *surely*.

Definition 8.13. The process W above is called a standard (one dimensional) Brownian motion.

Last time: $X_n \longrightarrow$ iid., $E X_n = \mu$, $Var(X_n) = \sigma^2$.

LLN: $S_n = \sum_1^n X_k$, $\frac{S_n}{n} \xrightarrow{n \to \infty} \mu$.

LLN: $\frac{(S_n - n\mu)}{n} \xrightarrow[a.s.]{n \to \infty} 0$ (LLN).

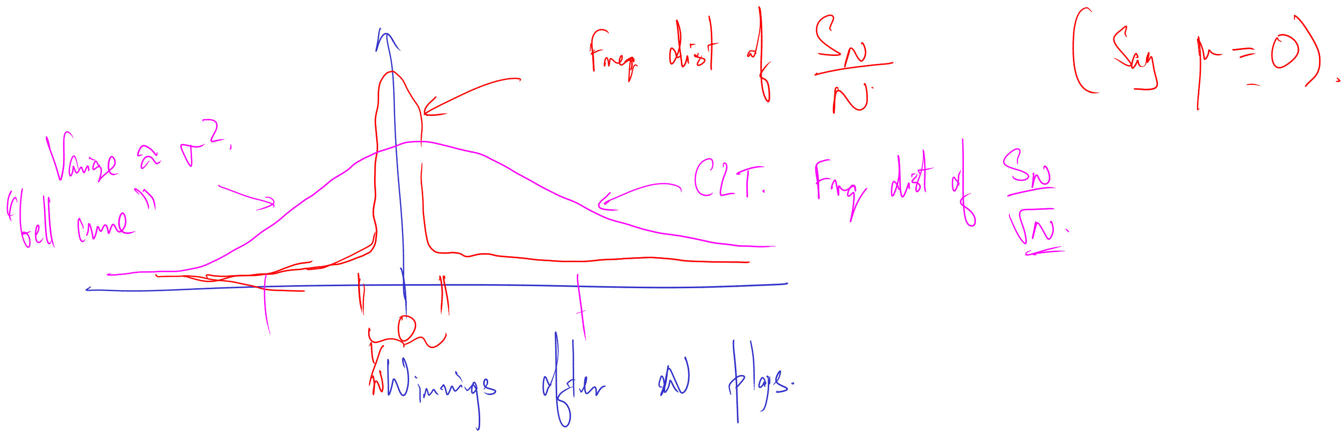
CLT: $\frac{(S_n - n\mu)}{\sqrt{n}} \xrightarrow{n \to \infty} N(0, \sigma^2)$

weak conv.

$$\left(\forall \text{ bdd cts } f,\ \lim_{n \to \infty} E f\left(\frac{S_n - n\mu}{\sqrt{n}}\right) = E f(N(0, \sigma^2))\right)$$

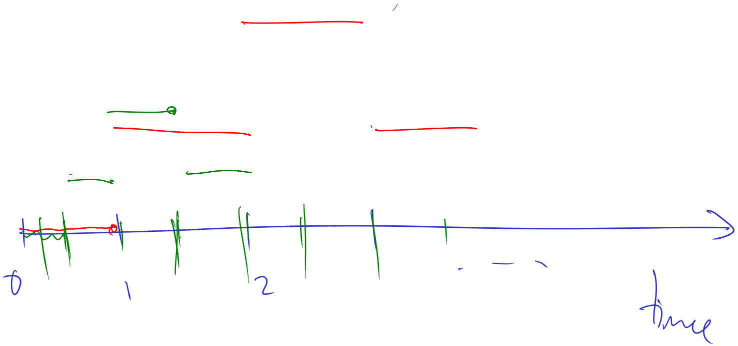
$$= \int_{-\infty}^{\infty} f(x)\, G_{\sigma^2}(x)\, dx$$

where $G_{\sigma^2}(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}\, dx$

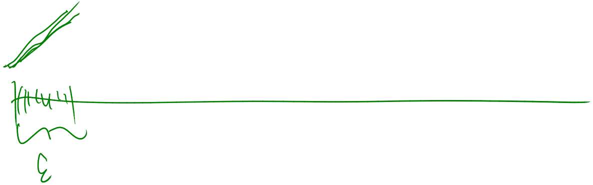


Brownian motion

"Cts time random walk"



Discrete RW.



8.3. Brownian motion.

- Suppose now $X_1, X_2, \dots$ are i.i.d. $\mathbb{R}$ valued random variables.
- Use $\tilde{\boldsymbol{P}}$ to denote the probability measure, and $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ to denote the associated expectation / conditional expectation.
- Assume $\tilde{\boldsymbol{E}} X_n = 0$, and $\tilde{\boldsymbol{E}} X_n^2 = 1$.

$(S_n = \sum_1^n X_k)$

Theorem 8.10. *Let $W_n^N = \frac{1}{\sqrt{N}} S_n = \frac{1}{\sqrt{N}} \sum_1^n X_k$. Then $\lim_{N \to \infty} W_{\lfloor Nt \rfloor}^N$ exists almost surely.*

(Subscript $\to$ time

Theorem 8.11. *(1) The function $t \mapsto W_t$ is continuous almost surely, and $W_0 = 0$.*
(2) If $0 = t_0 < t_1 < \cdots t_n$, then $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, ..., $W_{t_n} - W_{t_{n-1}}$ are independent and $\boxed{W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})}$.

Remark 8.12. Typically one changes the probability space to ensure the function $t \mapsto W_t$ is continuous $\boxed{\textit{surely}}$.

Definition 8.13. The process W above is called a standard (one dimensional) Brownian motion.

Let $W_t = \lim_{N \to \infty} W^N_{\lfloor Nt \rfloor}$

$\lfloor Nt \rfloor$ = largest int $\leq Nt$

= floor(Nt)

$W_t - W_s \sim N(0, t - s)$

The full proof of Theorems 8.10 and 8.11 are technical and beyond the scope of this course. However, we can prove a weaker result here:

Proposition 8.14. $W_T \sim \mathcal{N}(0, T)$.

Remark 8.15. The above is simply the central limit theorem (which we never proved). We will prove it here. Our proof can also be modified to prove that W has independent normally distributed increments.

Lemma 8.16. *Let f be a bounded continuous function, fix $T > 0$. By the Markov property we know $\tilde{\boldsymbol{E}}_n f(W^N_{\lfloor NT \rfloor}) = g_n(W^N_n)$ for some function g_n. Set $u(t,x) = \lim_{N\to\infty} g_{\lfloor Nt \rfloor}(x)$. Then $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ and $u(T,x) = f(x)$.*

Markov.

$$W^N_{\lfloor NT \rfloor} = \frac{1}{\sqrt{N}} \sum_{k=0}^{\lfloor NT \rfloor} X_k \leftarrow \text{Markov Process}$$

Pf: Note: Clearly $u(T,x) = \lim_{N\to\infty} \overbrace{g_{\lfloor NT \rfloor}(x)} = f(x)$

TOU: $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$

① Find a recurrence relation for g_n's

$$g_n(W_n^N) = \tilde{E}_n f\left(W^N_{\lfloor NT \rfloor}\right)$$

$$W_n^N = \frac{1}{\sqrt{N}} \sum_1^n X_k$$

$$= \tilde{E}_n \tilde{E}_{n+1} f\left(W^N_{\lfloor NT \rfloor}\right)$$

$$= \tilde{E}_n \, g_{n+1}\left(W^N_{n+1}\right) = \tilde{E}_n \, g_{n+1}\underbrace{\left(W_n^N + \frac{1}{\sqrt{N}} X_{n+1}\right)}$$

Note $W_n^N \longrightarrow \mathcal{F}_n$ meas

$\frac{1}{\sqrt{N}} X_{n+1} \longrightarrow$ indep of $\mathcal{F}_n$ $\Big\}$ — Indep lemma

& Say Range of $X_{n+1} = \{x_1, \dots x_m\}$ & $p_i = P(X_{n+1} = x_i)$.

Then $g_n(\underbrace{W_n^N}_{x}) = \tilde{E}_n\left(g_{n+1}\left(W_n^N + \frac{1}{\sqrt{N}} X_{n+1}\right)\right)$

$$\overset{\text{indep lemma}}{=} \sum_{i=1}^{M} p_i \, g_{n+1}\left(\underbrace{W_n^N}_{x_i} + \frac{x_i}{\sqrt{N}}\right)$$

Let $x = W_n^N$. $\Rightarrow$ $\boxed{g_n(x) = \sum_{i=1}^{m} g_{n+1}\left(x + \frac{x_i}{\sqrt{N}}\right) p_i}$

Taylor expand g_{n+1} : $\quad g_{n+1}(x+h) = g(x) + h\,g'(x) + \frac{1}{2}h^2 g''(x) + O(h^3)$

$$\hookrightarrow g_n(x) = \sum_{i=1}^{m} p_i\left[g_{n+1}(x) + \frac{x_i}{\sqrt{N}}\,g'(x) + \frac{1}{2N}\,x_i^2\,g''(x)\right] + O\left(\frac{1}{N^{3/2}}\right)$$

$$= g_{n+1}(x) + \frac{1}{\sqrt{N}}\underbrace{\left(\sum p_i x_i\right)}_{EX_{n+1}=0} g'(x) + \frac{1}{2N}\underbrace{\left(\sum x_i^2 p_i\right)}_{EX_{n+1}^2=1} g''(x).$$

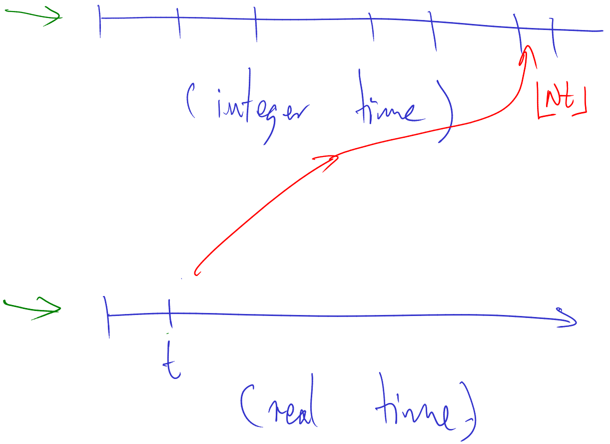
$$\Rightarrow \underbrace{\frac{g_n(x) - g_{n+1}(x)}{1/N}}_{\substack{\big\downarrow \ \text{(Next time)} \\ -\partial_t u(x)}} = 0 + \underbrace{\frac{1}{2} g''_{n+1}(x)}_{\substack{\big\downarrow \ \text{(Next time)} \\ \frac{1}{2}\partial_x^2 u(x)}} + \underbrace{O\left(\frac{1}{\sqrt{N}}\right)}_{\substack{\big\downarrow \\ 0}}$$

Last time: X_k (iid) $\tilde{E} X_k = 0$, $\tilde{E} X_k^2 = 1$

$$S_n = \sum_{k=1}^{n} X_k$$

$$W_n^N = \frac{1}{\sqrt{N}} S_n$$

Define $W_t = \lim_{N \to \infty} W^N_{\lfloor Nt \rfloor}$



Thm: Above lim exists a.s.

Thm: (1) $t \to W_t$ is cts as a fn of t.

(2) W_s is ind of $W_t - W_s$ & $W_t - W_s \sim N(0, t-s)$

Lemma 8.16. *Let f be a bounded continuous function, fix $T > 0$. By the Markov property we know $\tilde{\boldsymbol{E}}_n f(W^N_{\lfloor NT \rfloor}) = g_n(W^N_n)$ for some function g_n. Set $u(t,x) = \lim_{N\to\infty} g_{\lfloor Nt \rfloor}(x)$. Then $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ and $u(T,x) = f(x)$.*

Last time: (1) $u(T,x) = f(x)$

(2) Wrote recurrence relation for g_n's, use Taylor's thm & got

(*) ... $\dfrac{g_n(x) - g_{n+1}(x)}{1/N} = \frac{1}{2} g_{n+1}''(x) + O\left(\frac{1}{\sqrt{N}}\right)$

(3) $u(t,x) = \lim_{N\to\infty} g_{\lfloor Nt \rfloor}(x) \Rightarrow \partial_x^2 u = \lim_{N\to\infty} g''_{\lfloor Nt \rfloor}(x)$ ... (**)

(4) $u(t,x) = \lim_{N \to \infty} g_{\lfloor Nt \rfloor}(x) \quad \dots$

$$\Rightarrow u(t,x) \approx g_{\lfloor Nt \rfloor}(x) + \text{something small.}$$

$$\Rightarrow u\left(t + \frac{1}{N}, x\right) \approx g_{\lfloor N(t+\frac{1}{N}) \rfloor}(x) + \text{something small}$$

$$\approx g_{\lfloor Nt \rfloor + 1}(x) + \text{'' small.}$$

$$\Rightarrow \frac{u(t,x) - u(t+\frac{1}{N}, x)}{1/N} \approx \left(\frac{g_{\lfloor Nt \rfloor}(x) - g_{\lfloor Nt \rfloor + 1}(x)}{1/N} \right) = \underbrace{\frac{1}{2} g''_{\lfloor Nt \rfloor + 1}(x)} + O\left(\frac{1}{\sqrt{N}}\right)$$

$$(**) \approx \frac{1}{2} \partial_x^2 u(t,x)$$

$$\Rightarrow \quad \underbrace{\frac{u(t,x) - u\left(t + \frac{1}{N}, x\right)}{1/N}} = \frac{1}{2} \partial_x^2 u(t,x) + O\left(\frac{1}{\sqrt{N}}\right)$$

$$\boxed{-\partial_t u(t,x) = \frac{1}{2} \partial_x u(t,x)}$$

QED.

Lemma 8.17. *Suppose $u = u(t,x)$ satisfies $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ for $t < T$ and $u(T,x) = f(x)$, then*

$$u(t,x) = \int_{\mathbb{R}} f(y) G_{T-t}(x-y)\,dy = \int_{\mathbb{R}} f(x-y) G_{T-t}(y)\,dy\,, \qquad \text{where } G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\,.$$

Density of Normal dist mean 0 & variance t

Pf: Start with the formula for u.

check $\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ & $u(T,x) = f$

① $\partial_x G_t(x) = \dfrac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \cdot \left(\dfrac{-x}{t}\right)$

② $\dfrac{1}{2}\partial_x^2 G_t(x) = \dfrac{1}{2\sqrt{2\pi t}} e^{-x^2/2t} \cdot \left(\dfrac{x^2}{t^3}\right) + \dfrac{1}{2\sqrt{2\pi t}} e^{-x^2/2t} \left(\dfrac{-1}{t}\right)$

③ $\partial_t G = \partial_t \left(\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \right) = -\frac{1}{2\sqrt{2\pi t^3}} \cdot e^{-x^2/2t} + \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \left(\frac{x^2}{2t^2} \right)$

$\Rightarrow$ ④ $\partial_t G_t = \frac{1}{2} \partial_x^2 G_t$.

$\Rightarrow$ ⑤ $\partial_t (G_{T-t}) + \frac{1}{2} \partial_x^2 G_{T-t} = 0$

$\Rightarrow$ ⑥. $u(t,x) = \int_R f(y)\, G_{T-t}(x-y)\, dy$

$$\Rightarrow \left(\partial_t + \frac{1}{2}\partial_x^2\right) u(t,x) = \left(\partial_t + \frac{1}{2}\partial_x^2\right) \int_{\mathbb{R}} f(y)\, G_{T-t}(x-y)\, dy$$

$$= \int_{\mathbb{R}} f(y) \underbrace{\left(\partial_t + \frac{1}{2}\partial_x^2\right) G_{T-t}(x-y)}_{=0}\, dy$$

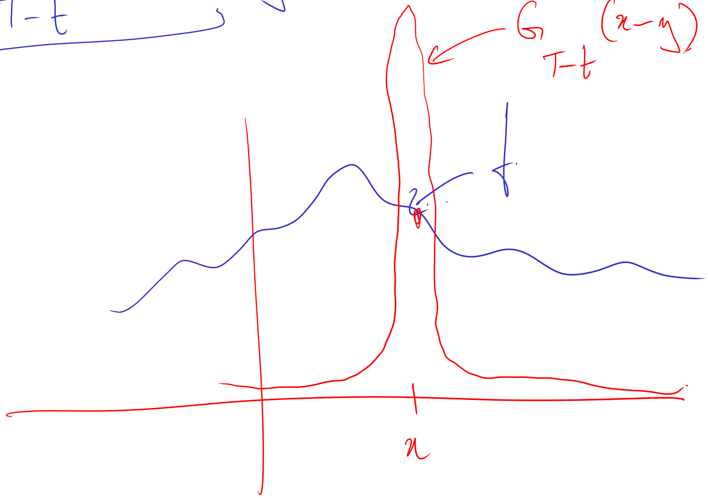
$$\Rightarrow \partial_t u + \frac{1}{2}\partial_x^2 u = 0$$

(7) Also NTC $u(x,T) = f(T)$.

$$u(x,t) = \int f(y)\, \underbrace{G_{T-t}(x-y)}\, dy$$

(t close to T)

$$\xrightarrow[t \to T]{\text{Exercise}} f(x)$$



QED

Proof of Proposition 8.14

Pf. $W_T \sim N(0, T)$.

Enough to show $E f(W_T) = \int f(x)\,\left(\text{PDf of } N(0,T)\right)\,dx$

$\forall$ bdd cts fns f.

Note: $u(0,0) = E f(W_T)$ where u is the fn from the above lemma.

Also know $u(0,0) = \displaystyle\int_{-\infty}^{\infty} f(y)\, G_{T-0}(0-y)\,dy = \int_{-\infty}^{\infty} f(y)\, \frac{e^{-y^2/2t}}{\sqrt{2\pi t}}\,dy$

PDF of $N(0,t)$

QED.

Definition 8.18. We say a random variable Y is $\boxed{\mathcal{F}_t \text{ measurable}}$ if $Y = \lim_{n\to\infty} f_n(W_{t_1}, \dots, W_{t_n})$ where $\boxed{t_i \leqslant t}$ for all i.

Definition 8.19. If $Y = f(W_{t_1}, \dots, W_{t_n})$ for some function f and $0 \leqslant t_1 \cdots < t_n$, define $\boxed{\boldsymbol{E}_t Y} = \underbrace{\lim_{N\to\infty} \boxed{\tilde{\boldsymbol{E}}_{\lfloor Nt \rfloor}} f(W^N_{\lfloor Nt_1 \rfloor}, \dots, W^N_{\lfloor Nt_n \rfloor})}$

Remark 8.20. $\boldsymbol{E}_t f(W_T) = u(t, W_t)$, where u is the function in Lemma 8.16.

Proposition 8.21. *W is a martingale.*

(dont need $t_i \le t$)

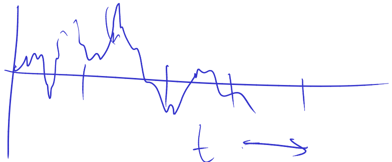
Disc time mg: $\quad E_m M_n = M_m \quad \forall\, m \le n.$

cts time $\quad E_s M_t = M_s$

2

Last time: $W_t = \lim_{N\to\infty} W^N_{\lfloor Nt \rfloor}$ ← Brownian motion.

$EX_k = 0$, $EX_k^2 = 1$, $X_k \to$ iid. $= \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{1}^{\lfloor Nt \rfloor} X_k$.



$t \in \mathbb{R}$.

Definition 8.18. We say a random variable Y is $\mathcal{F}_t$ measurable if $Y = \lim_{n\to\infty} f_n(W_{t_1}, \dots, W_{t_n})$ where $t_i \leqslant t$ for all i.

Definition 8.19. If $Y = f(W_{t_1}, \dots, W_{t_n})$ for some function f and $0 \leqslant t_1 \cdots < t_n$, define $\boldsymbol{E}_t Y = \lim_{N\to\infty} \tilde{\boldsymbol{E}}_{\lfloor Nt \rfloor} f(W^N_{\lfloor Nt_1 \rfloor}, \dots, W^N_{\lfloor Nt_n \rfloor})$

Remark 8.20. $\boldsymbol{E}_t f(W_T) = u(t, W_t)$, where u is the function in Lemma 8.16.

Remark 8.21. The operator $\boldsymbol{E}_t$ satisfies the same properties as $\tilde{\boldsymbol{E}}_n$ (e.g. $\boldsymbol{E}_t(XY) = X\boldsymbol{E}_t Y$ if X is $\mathcal{F}_t$ measurable, independence lemma, etc.) These will be developed systematically in continuous time finance.

Proposition 8.22. *W is a martingale.*

$E_t Y$ = conditional exp of Y given $\mathcal{F}_t$

$= E(Y \mid \mathcal{F}_t)$.

W^N, S_n, X_n etc

(int time)

$\lfloor Nt \rfloor$

W

$(t \in \mathbb{R})$

E_t has the same properties as $\tilde{E}_n$ in the disc case

① X is $\mathcal{F}_t$ meas $\Rightarrow$ $E_t f(X) = f(X)$

② X is $\mathcal{F}_t$ meas & $\underline{Y \text{ is any thing}}$ $\Rightarrow$ $E_t(XY) = \underline{X E_t Y}$

③ X ind of $\mathcal{F}_t$ $\Rightarrow$ $E_t X = EX$

Prop: W is a Mg

Q: What does it mean for a c/s time process to be a mg

Disc time def: M is a (disc time) mg if $\tilde{E}_n M_{n+1} = M_n$ $\forall n$.

$\rightarrow$ C/s time guess ⓪: $E_t(M_{t+1}) = M_t$ $\forall t$ (Not good enough).

Def: We say M is a mg (cts time) if M is adapted and $\underline{E_s M_t = M_s}$ $\forall\, t \geq s$.

(Note: disc time: by tower prop & ind, $E_{n} M_{n+1} = M_n \;\forall\, n$

$\iff E_m M_{\underline{n}} = \underline{M_m} \;\forall\, n \geq m$)

<u>Prop:</u> W is a mg.

Pf: WTS $E_s W_t = W_s$ $\forall\, t \geq s$.

Note: $E_s W_t = E_s(W_t - W_s + W_s)$

$\boxed{\text{Trick \# 1 for B.M}}$

$= E_s(W_t - W_s) + E_s W_s$

$= E(W_t - W_s) + W_s$ $\quad$ ($\because W_t - W_s$ is ind of $\mathcal{F}_s$ & $W_t - W_s \sim N(0, t-s)$)

$= 0 + W_s = W_s$ $\quad$ ($\because W_s$ is $\mathcal{F}_s$ meas)

QED.

8.4. Convergence of the Binomial Model.

(1) Let $r_N > -1$, and consider a bank that pays you interest r_N every $1/N$ time units.
(2) *Question:* Can we choose r_N so that this converges as $N \to \infty$.
(3) Let $C_0^N = 1$, $C_{n+1}^N = (1 + r_N)C_n^N$ and $C_t = \lim_{N\to\infty} C_{\lfloor Nt \rfloor}^N$.

C^N

(int time).

$t \mapsto \lfloor Nt \rfloor$

Proposition 8.23. *If $r \in \mathbb{R}$, $r_N = r/N$, then $C_t = e^{rt}$.*

Remark 8.24. Note $\partial_t C_t = rC_t$. The quantity r is known as the *continuously compounded interest rate.*

$t \in \mathbb{R}$

Remark 8.25. If the interest rate is a constant r, then the discount factor is simply $\boxed{D_t = 1/C_t = e^{-rt}}$.

Pf: $r_N = \dfrac{r}{N}$ $\quad r \in \mathbb{R}$.

$$C_n^N = (1 + r_N)^n = \left(1 + \frac{r}{N}\right)^n.$$

$$\Rightarrow C_{\lfloor Nt \rfloor}^N = \left(1 + \frac{r}{N}\right)^{\lfloor Nt \rfloor} = \Bigg[\underbrace{\left(1 + \frac{r}{N}\right)^N}\Bigg]^{\frac{\lfloor Nt \rfloor}{N}}$$

Note $\lim_{N\to\infty} \left(1 + \dfrac{r}{N}\right)^N = e^r$ & $\lim_{N\to\infty} \dfrac{\lfloor Nt \rfloor}{N} = t$

$$\Rightarrow \lim_{N \to \infty} C^{N}_{\lfloor Nt \rfloor} = e^{rt}$$

QED

(1) Now consider the N period Binomial model, with parameters $0 < d_N < 1 + r_N < u_N$, with stock price denoted by S_n^N.
(2) Each time step for S^N denotes $1/N$ time units in real time. Can we chose u_N, d_N, r_N such that $S_t = \lim_{N\to\infty} S^N_{\lfloor Nt \rfloor}$ exists?
(3) Choose $r_N = r/N$, where $r \in \mathbb{R}$ is the continuously compounded interest rate.

Theorem 8.26. *Let $u, d > 0$ and choose*

doesnt dep on N.

$$u_N = 1 + \frac{r}{N} + \frac{u}{\sqrt{N}}, \qquad d_N = 1 + \frac{r}{N} - \frac{d}{\sqrt{N}}, \qquad \tilde{p} = \frac{d}{u+d}, \qquad \tilde{q} = \frac{u}{u+d}, \qquad \boxed{\sigma^2 = \tilde{p}u^2 + \tilde{q}d^2}.$$

Under the risk neutral measure, the processes $S^N_{\lfloor Nt \rfloor}$ converge weakly to $S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}$, where W is a Brownian motion. That is, for any bounded continuous function f,

GBM

$$\lim_{N\to\infty} \tilde{\boldsymbol{E}} f(S^N_{\lfloor Nt \rfloor}) = \tilde{\boldsymbol{E}} f(S_t) = \tilde{\boldsymbol{E}} f\Big(S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)\Big)$$

Remark 8.27. S_t above is called a *Geometric Brownian motion* with mean return rate r, and volatility σ.

Remark 8.28. The fact that we took the limit under the risk neutral measure is the reason the mean return rate r is the same as the interest rate r.

Remark 8.29. In this continuous time market you have the asset (whose price is denoted by S_t), and a bank with *continuously compounded* interest rate r (i.e. discount factor is $D_t = e^{-rt}$). You can trade *continuously* in time, and we are neglecting any transaction costs.

R.N. Prob.

$$\tilde{p}_N = \frac{1+r_N-d_N}{u_N-d_N} = \frac{1+\frac{r}{N}-\left(1+\frac{r}{N}-\frac{d}{\sqrt{N}}\right)}{\left(1+\frac{r}{N}+\frac{u}{\sqrt{N}}\right)-\left(1+\frac{r}{N}-\frac{d}{\sqrt{N}}\right)}$$

$$= \frac{d/\cancel{\sqrt{N}}}{(u+d)/\cancel{\sqrt{N}}} = \frac{d}{u+d}.$$

Theorem 8.30. *Consider a security that pays $f(S_T)$ at maturity time T. The arbitrage free price of this security at time t is given by*

$$\boxed{V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t\Big(D_T f(S_T)\Big) = \tilde{\boldsymbol{E}}_t\big(e^{-r(T-t)} f(S_T)\big)}$$

Proof. For the Binomial model we already know $V_n^N = \frac{1}{D_n^N}\tilde{\boldsymbol{E}}_n D_{\lfloor NT\rfloor}^N f(S_{\lfloor NT\rfloor}^N)$. Set $n = \lfloor Nt\rfloor$ and send $N \to \infty$. $\square$

8.4. **Convergence of the Binomial Model.**

(1) Let $r_N > -1$, and consider a bank that pays you interest r_N every $1/N$ time units.
(2) *Question:* Can we choose r_N so that this converges as $N \to \infty$.
(3) Let $C_0^N = 1$, $C_{n+1}^N = (1 + r_N)C_n^N$ and $C_t = \lim_{N\to\infty} C_{\lfloor Nt \rfloor}^N$.

Proposition 8.23. *If* $r \in \mathbb{R}$, $r_N = r/N$, *then* $C_t = e^{rt}$.

Remark 8.24. Note $\partial_t C_t = rC_t$. The quantity r is known as the *continuously compounded interest rate.*

Remark 8.25. If the interest rate is a constant r, then the discount factor is simply $D_t = 1/C_t = e^{-rt}$.

$\to \lfloor Nt \rfloor$

(1) Now consider the N period Binomial model, with parameters $0 < d_N < 1 + r_N < u_N$, with stock price denoted by S_n^N.
(2) Each time step for S^N denotes $1/N$ time units in real time. Can we chose u_N, d_N, r_N such that $S_t = \lim_{N\to\infty} S^N_{\lfloor Nt \rfloor}$ exists?
(3) Choose $r_N = r/N$, where $r \in \mathbb{R}$ is the continuously compounded interest rate.

Theorem 8.26. *Let $u, d > 0$ and choose*

$$u_N = 1 + \frac{r}{N} + \frac{u}{\sqrt{N}}\,, \qquad d_N = 1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\,, \qquad \tilde{p} = \frac{d}{u+d}\,, \qquad \tilde{q} = \frac{u}{u+d}\,, \qquad \sigma^2 = \tilde{p}u^2 + \tilde{q}d^2\,.$$

Under the risk neutral measure, the processes $S^N_{\lfloor Nt \rfloor}$ converge weakly to $S_t = S_0 e^{(r-\sigma^2/2)t + \sigma W_t}$, where W is a Brownian motion. That is, for any bounded continuous function f,

$$\lim_{N\to\infty} \tilde{\boldsymbol{E}} f(S^N_{\lfloor Nt \rfloor}) = \tilde{\boldsymbol{E}} f(S_t) = \tilde{\boldsymbol{E}} f\Big(S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)\Big)$$

Remark 8.27. S_t above is called a *Geometric Brownian motion* with mean return rate r, and volatility σ.

Remark 8.28. The fact that we took the limit under the risk neutral measure is the reason the mean return rate r is the same as the interest rate r.

Remark 8.29. In this continuous time market you have the asset (whose price is denoted by S_t), and a bank with *continuously compounded* interest rate r (i.e. discount factor is $D_t = e^{-rt}$). You can trade *continuously* in time, and we are neglecting any transaction costs.

u

Note : RN prob in the N period case and

$$\tilde{p}_N = \frac{1 + r_N - d_N}{u_N - d_N} = \frac{1 + \frac{r}{N} - \left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right)}{\left(1 + \frac{r}{N} + \frac{u}{\sqrt{N}}\right) - \left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right)}$$

$$= \frac{d/\sqrt{N}}{(u+d)/\sqrt{N}} = \frac{d}{u+d}.$$

n

Pf: $S^N_{n+1} = \begin{cases} u_N S^N_n & \text{with prob } \tilde{p} = \frac{d}{u+d} \\ d_N S^N_n & \text{with prob } \tilde{q} = \frac{u}{u+d}. \end{cases}$

Let $Y^N_{n+1} = \ln\left(\frac{S^N_{n+1}}{S^N_n}\right) \Rightarrow S^N_{n+1} = S^N_n \cdot e^{Y^N_{n+1}}$

$\Rightarrow S^N_{n+1} = S^N_{n-1}\, e^{Y^N_{n+1} + Y^N_n} \dots$ $\boxed{S^N_n = S_0 \cdot e^{\sum_1^n Y_k}}$

② Know if $EX_n = 0$, $EX_n^2 = 1$, iid, $\frac{1}{\sqrt{N}} \sum_1^{\lfloor Nt \rfloor} X_k \longrightarrow W_t$ (B.M.)

③ Compute $\mu_N = E\, Y_k^N \overset{iid}{=} E\, Y_1^N$

& $\sigma_N^2 = Var(Y_k^N) \overset{iid}{=} Var(Y_1^N)$.

ⓐ $\mu_N : E\, Y_k^N = \tilde{p} \ln u_N + \tilde{q} \ln d_N$

$$= \tilde{p} \ln\left(1 + \frac{r}{N} + \frac{u}{\sqrt{N}}\right) + \tilde{q} \ln\left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right) \ldots (*)$$

ⓑ Taylor expand the fn $\ln(1+x)$:

$$\ln(1+x) = 0 + x + \frac{1}{2}(-1)x^2 + O(x^3).$$

$$\boxed{\ln(1+x) \approx x - \frac{x^2}{2} + O(x^3)}$$

(c) Use in (*): $\mu_N = E\, Y_k^N = \tilde{p} \ln\left(1 + \frac{r}{N} + \frac{u}{\sqrt{N}}\right) + \tilde{q} \ln\left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right)$

$$= \frac{1}{\sqrt{N}}\left(\tilde{p}\, u - \tilde{q}\, d\right) + \frac{1}{N}\left(\tilde{p}\, r + \tilde{q}\, r - \frac{1}{2}\left(\tilde{p}\, u^2 + \tilde{q}\, d^2\right)\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$= \frac{1}{\sqrt{N}}\underbrace{\left(\frac{d}{u+d} \cdot u - \frac{u}{u+d}\, d\right)}_{0} + \frac{1}{N}\left(r - \frac{\sigma^2}{2}\right)$$

③ You compute $\sigma_N^2 = \mathrm{Var}(Y^N_k) = \frac{\sigma^2}{N} + O\left(\frac{1}{N^{3/2}}\right)$

④ Set $X^N_n = \frac{Y^N_n - \mu_N}{\sigma_N} \iff Y^N_n = \mu_N + \sigma_N X^N_n$.

Note $E X^N_n = 0$ & $\mathrm{Var}(X^N_n) = 1$.

⑤ $\Rightarrow \sum_1^n Y^N_k = \sigma_N \sum_1^n X_k + n\mu_N$

$$= \frac{\sigma}{\sqrt{N}} \sum_{1}^{n} X_k + \frac{n}{N}\left(r - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{N^{3/2}}\right)$$

(6) B.M. constr $\Rightarrow$ $$\lim_{N\to\infty} \sum_{1}^{\lfloor Nt \rfloor} Y_k = \lim_{N\to\infty}\left(\frac{\sigma}{\sqrt{N}} \sum_{1}^{\lfloor Nt \rfloor} X_k + \frac{\lfloor Nt \rfloor}{N}\left(r - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{N}}\right)\right)$$

$$= \sigma W_t + t\left(r - \frac{\sigma^2}{2}\right)$$

(7) $\Rightarrow$ $$\lim_{N\to\infty} S^N_{\lfloor Nt \rfloor} = \lim_{N\to\infty} S_0 e^{\sum_{1}^{\lfloor Nt \rfloor} Y_k} = \underline{S_0 \exp\left(t\left(r - \frac{\sigma^2}{2}\right) + \sigma W_t\right)}$$ QED

Theorem 8.30. *Consider a security that pays $f(S_T)$ at maturity time T. The arbitrage free price of this security at time t is given by*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t\Big(D_T f(S_T)\Big) = \tilde{\boldsymbol{E}}_t\big(e^{-r(T-t)}f(S_T)\big)$$

Proof. For the Binomial model we already know $V_n^N = \frac{1}{D_n^N}\tilde{\boldsymbol{E}}_n D^N_{\lfloor NT\rfloor} f(S^N_{\lfloor NT\rfloor})$. Set $n = \lfloor Nt\rfloor$ and send $N \to \infty$. □

(Last time)

Proof of Theorem 8.26

Did alone

Theorem 8.31 (Black-Scholes formula). *In the above market, a European call with maturity T and strike K pays $(S_T - K)^+$ at time T. The arbitrage free price of this call at time t is $c(t, S_t)$, where*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))\,,$$

$$\textit{where}\quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,.$$

← CDF of norml.

Proof. Let $\tau = T - t$. We know $\boxed{c(t, S(t)) = \tilde{\boldsymbol{E}}_t e^{-r\tau}(S_T - K)^+.}$ Observe first

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma W_t}\,, \qquad S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T}\,, \qquad \implies \qquad S_T = S_t e^{(r-\frac{\sigma^2}{2})\tau + \sigma(W_T - W_t)}\,,$$

Since $W_T - W_t$ is independent of $\mathcal{F}_t$, and S_t is $\mathcal{F}_t$ measurable, by the independence lemma,

$$c(t, S_t) = \tilde{\boldsymbol{E}}_t e^{-r\tau}\big(S_t e^{(r-\frac{\sigma^2}{2})\tau+\sigma(W_T - W_t)} - K\big)^+ = \int_{\mathbb{R}} e^{-r\tau}\big(S_t e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y} - K\big)^+ e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}\,.$$

← PDF of std norml.

Now set $S_t = x$,

$$d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \quad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy = \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-y^2/2}\,dy\,,$$

and observe

$$\begin{aligned} c(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} x\exp\Big(\frac{-\sigma^2\tau}{2} + \sigma\sqrt{\tau}y - \frac{y^2}{2}\Big)\,dy - e^{-r\tau}KN(d_-) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} x\exp\Big(\frac{-(y-\sigma\sqrt{\tau})^2}{2}\Big)\,dy - e^{-r\tau}KN(d_-) = \boxed{xN(d_+) - e^{-r\tau}KN(d_-)}. \end{aligned}$$

□

$$W_T - W_t \sim N(0, T-t)$$

$$W_T - W_t = \sqrt{\tau} \cdot \underbrace{\left(\frac{W_T - W_t}{\sqrt{\tau}} \right)}_{N(0,1)}$$

$$\Rightarrow \frac{W_T - W_t}{\sqrt{T-t}} \sim N(0, 1)$$

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① Please fill FCE's ; 75% respond rate before final

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